# Non-Linear Models Cont'd: Infectious Diseases

Infectious Diseases

- Can be classified into 2 broad categories:
  1. those caused by viruses & bacteria (*microparasitic* diseases e.g. smallpox, measles),
  2. those due to 'vectors' (*macroparasitic* diseases such as malaria).
- Main distinction btw them:
  1. former reproduce within the host & are transmitted directly from one host to another,
  2. latter require some means of transmission (i.e. insect-/ entertainment-borne etc.).
- We focus on microparasitic diseases.



Notes



# Non-Linear Models Cont'd: Infectious Diseases

- Virus & host organism are not a predator-prey system, for a number of reasons:
  1. Virus numbers in individual host vary greatly, thus model can't say how many are infected.
  2. Predator-prey model does not have any stable equilibria.
  3. Key question is disease spread in *population*, difficult to model with Predator-prey.
- To model diseases, pop'n divided into 3 groups:
  1. *suscepibles* (i.e. those not immune to disease),
  2. *infectives* (those who can infect non-immunes),
  3. *removed* (dead or in quarantine or immune).
- Symbols $S, I, R$ denote these resp groups at time $t$.



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- Following assumptions are made:
  1. rate of change of infectives $\propto$ number of contacts btw $I$ & $S$ (or each $I$ infects a constant fraction $\beta$ of $S$ per unit time)
  2. the number of $I$ who become $R$ is proportional to $I$
- Equations are thus:

$$\begin{aligned} \dot{S} &= -\beta IS \\ \dot{I} &= \beta IS - \nu I \\ \dot{R} &= \nu I \end{aligned} \qquad (5.30)$$

$\beta, \nu$ are *infection* & *removal rates*, $\dot{S} = \frac{dS}{dt}$ etc.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- $\beta$ governs speed of disease spread
- $\nu$ governs rate infected hosts die/otherwise removed.
- By adding 3 equations, can see that

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0,$$

  i.e. $S + I + R = N$ for constant (initial) population $N$.
- Rest of the initial conditions are
  $S(0) = S_0 > 0,\ I(0) = I_0 > 0,\ R(0) = 0.0$
- Given these initial conditions from Eqn.(5.30b) get:

$$\left.\frac{dI}{dt}\right|_{t=0} = I_0\left(\beta S_0 - \nu\right) \begin{cases} < 0, & \text{if } S_0 < \nu/\beta \\ > 0, & \text{if } S_0 > \nu/\beta \end{cases} \qquad (5.31)$$



Notes

# Non-Linear Models Cont'd: Infectious Diseases

- Ratio $\rho = \nu/\beta$, known as *relative* removal rate.
- From Eqn.(5.30a), $\frac{dS}{dt} < 0$
- So $S(t) \leq S_0$ & (if $S_0 < \rho$) so $S(t) < \rho$ for all $t$.
- Corresponding to different values of $\rho$, we have two distinct cases
  1. From Eqn.(5.30b), if $S < \rho$
     - Then $\frac{dI}{dt} < 0$ meaning infectives never grow.
     - So as $t \to \infty$, infection will die out.
  2. If $S_0 > \rho$,
     - In same way $\frac{dI}{dt} > 0$ for all $t$ such that $S(t) > \rho$.
     - Implies for some time $t \in [0, t_0)$, must have $I(t) > I_0$.
     - Say this is an *epidemic* situation.



Notes



# Non-Linear Models Cont'd: Infectious Diseases

- A plot of $S$, $I$ and $R$ is shown in Fig 5.6.

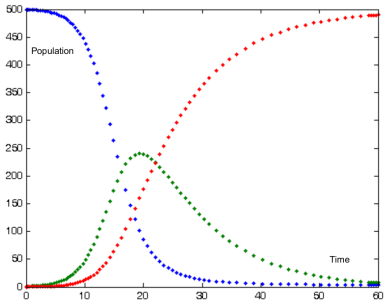



FIGURE 5.6 : SIR Model for $\rho = 100$: Susceptable, Infected, Removed



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- To further analyse the model, see $(S, I)$ phase-plane in Fig. 5.7.
- Can express Eqn.s(5.30a,b) as

$$\frac{dI}{dS} = \frac{I(\beta S - \nu)}{-\beta I S} = -1 + \frac{\rho}{S}, \tag{5.32}$$

provided $I \neq 0$. Can derive:

$$I(t) = I_0 - S(t) + \rho \ln S(t) + S_0 - \rho \ln S_0 \tag{5.33}$$

- See from Eqn.(5.33), Fig. 5.7 that $I_{\max}$ when $S = \rho$.
- Show trajectories various $(S_0, I_0)$ &, as per predator-prey, all move anti-clockwise.
- A trajectory starts on line $I + S = N$ (as $R(0) = 0$ & susceptibles decrease with time.
- If $(S_0, I_0)$ is in epidemic region (right of $\rho = 4$ in Fig. 5.7), $I \uparrow$ until $S = \rho$.



Notes

Intro to the Topic
Time Series
Discrete Models
Growth and Decay
Linear & Non-Linear Interaction Models
Linear Models
Non-Linear Models

## Non-Linear Models Cont'd: Infectious Diseases

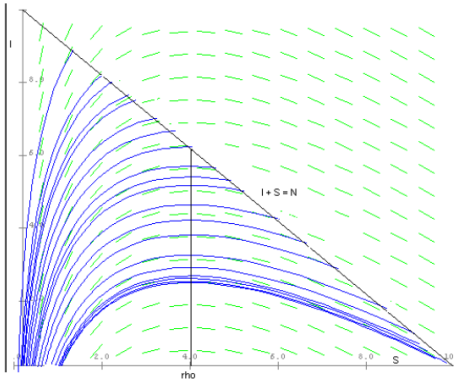



FIGURE 5.7 : SIR Model Phase-Plane Plot for $\rho = 4$



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- How does disease eventually die out with SIR model?
- From Eqn.s(5.30a,c):

$$\frac{dS}{dR} = \frac{-\beta IS}{\nu I} = -\frac{S}{\rho}, \qquad (5.34)$$

thus $S = S_0 e^{-R/\rho}$. But, $\forall t \geq 0, R \leq N$ this means$S(t) \geq S_0 e^{-N/\rho}$, i.e. as $t \to \infty$, $S > 0$.

- But, from Eqn.(5.30c) $\frac{dR}{dt} > 0 \ \ \forall t \geq 0$, so, if need to keep $R \leq N$, then need $\frac{dR}{dt} \to 0$, as $t \to \infty$.
- From Eqn.(5.30c), this can only happen if $I(\infty) = 0$.
- Hence lack of infectives wipe out disease & not lack of susceptibles.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- More general model than SIR (more applicable), is one with only partial immunity.
- Called SIRS model & permits previously infected (i.e. removed) individuals to return to susceptible pop'n
- Do so at a rate proportional to the number removed.
- Mathematically SIRS can be expressed as:

$$\begin{aligned} \frac{dS}{dt} &= -\beta IS & +\gamma R \\ \frac{dI}{dt} &= \beta IS & -\nu I \\ \frac{dR}{dt} &= & \nu I \quad -\gamma R \end{aligned} \qquad (5.35)$$

- Again the total population, $S + I + R = N$, is constant.



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- SIRS can be analysed using standard methods, equilibrium states found to be:

$$\begin{aligned} \dot{S} = 0 \quad &\Rightarrow \quad \frac{\beta IS}{\gamma} = R \\ \dot{I} = 0 \quad &\Rightarrow \quad \beta IS = \nu I \\ \dot{R} = 0 \quad &\Rightarrow \quad \frac{\nu I}{\gamma} = R \end{aligned} \tag{5.36}$$

- These yield 2 equilibrium points:
  1. first is trivial $\bar{S}_1 = N, \quad \bar{I}_1 = 0 \quad \bar{R}_1 = 0$,i.e. all pop'n healthy but susceptible & disease eradicated;
- This equilibrium point can be shown to be stable.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- A further equilibrium point can be found:
  2. Put $S = \nu/\beta$ (from Eqn.(5.36b)) & $R = \nu I/\gamma$ (from Eqn.(5.36c)) into $S + I + R = N$ to give:

$$\bar{S}_2 = \frac{\nu}{\beta}, \quad \bar{I}_2 = \frac{\gamma}{\beta}\frac{\beta N - \nu}{\gamma + \nu}, \quad \bar{R}_2 = \frac{\nu}{\beta}\frac{\beta N - \nu}{\gamma + \nu} \tag{5.37}$$

- $(\bar{S}_2, \bar{I}_2, \bar{R}_2)$ is only meaningful if all values are +ive.
- i.e. $\frac{\beta}{\nu} N > 1$ (i.e. +ive numerators for $\bar{I}_2, \bar{R}_2$).
- This *threshold effect* is minimum population necessary for a disease to become endemic.



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- Have seen $\rho = \nu/\beta$ (*relative removal rate*) above;
- Now define its reciprocal $\beta/\nu$ as follows:
  - As removal rate from infectives is $\nu$ (units 1/time), so average infectivity period is $1/\nu$.
  - $\beta$ is fraction of contacts (between $I$ & $S$) resulting in infections,
  - So $\beta \times 1/\nu$ is population fraction coming into contact with $I$s during infectious period.
- Hence $\sigma = \beta N/\nu$ is disease's *infectious contact number* or *intrinsic reproducive rate* sometimes denoted $R_0$.
- So, from above, and usefully enough, the disease will become endemic in the population if $\sigma > 1$.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- Can see this effect quite clearly in phase-plane.
- By using $R = N - S - I$, can write Eqn.s(5.35a,b) as:

$$\begin{array}{rcl} \frac{dS}{dt} & = & -\beta IS + \gamma(N - S - I) \\ \frac{dI}{dt} & = & \beta IS - \nu I \end{array} \qquad (5.38)$$

- In Fig 8(a) pop'n cannot sustain disease & it dies out;
- In Fig 8(b), get steady-state

$$\bar{S}_2 = \frac{\nu}{\beta}, \quad \bar{I}_2 = \frac{\gamma}{\beta}\frac{\beta N - \nu}{\gamma + \nu}.$$

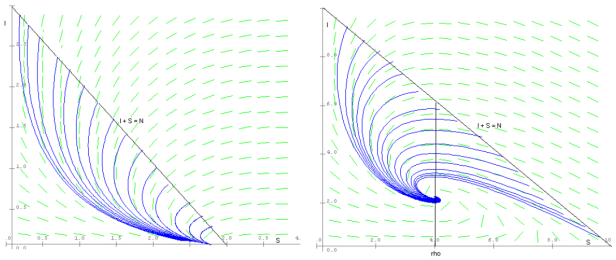
- Jacobian corresponding to Eqn.(5.38) at $(\bar{S}_2, \bar{I}_2)$ is

$$\mathbf{A}(\bar{S}_2, \bar{I}_2) \quad = \quad \begin{pmatrix} -(\beta\bar{I}_2 + \gamma) & -(\beta\bar{S}_2 + \gamma) \\ -\beta\bar{I}_2 & \beta\bar{S}_2 - \nu \end{pmatrix} \qquad (5.39)$$



Notes

## Non-Linear Models Cont'd: Infectious Diseases



(a) $(S, I)$ for $\sigma < 1$ (b) $(S, I)$ for $\sigma > 1$

FIGURE 5.8 : SIRS Phase-Plane Plots

- From above, stability conditions are: trace of Jacobian is negative & determinant, positive.
- Can show that equilibrium point $(\bar{S}_2, \bar{I}_2)$ is stable when $\sigma > 1$ is satisfied.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

The SIS Model:

- A special case of SIR where infection doesn't give lasting immunity.
- Such infections[5] have no recovered state & individuals become susceptible again after infection.
- The equations are thus:

$$\begin{aligned} \frac{dS}{dt} &= -\beta IS + \nu I \\ \frac{dI}{dt} &= \beta IS - \nu I \end{aligned} \tag{5.40}$$

[5] e.g. tuberculosis, meningitis, & infections leading to the common cold



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- So for total population, $N$, it holds that

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} = 0$$

  i.e. $S + I = N$ for constant (initial) population $N$.
- Expressing $I$ in terms of $S$ in eqn.(5.40), can be seen that:

$$\frac{dI}{dt} = (\beta N - \nu)I - \beta I^2$$

- This is a form of the logistic growth equation with $r = \beta N - \nu$ and $K = N - \frac{\nu}{\beta}$ so that we have two cases:
  1. for $\frac{\beta}{\nu}N > 1$, $\lim_{t\to\infty} I(t) = \frac{\beta N - \nu}{\beta}$ & disease will spread,
  2. for $\frac{\beta}{\nu}N \leq 1$, $\lim_{t\to\infty} I(t) = 0$ & disease will die out.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- A plot of the former SIS model case is shown in Fig 5.9.

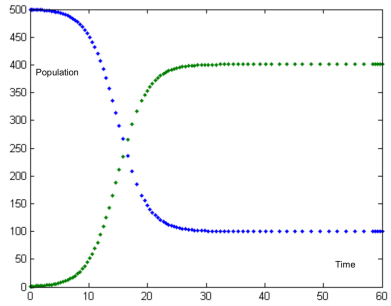



FIGURE 5.9 : SIS Model for $\rho = 100$: Susceptable, Infected



Notes

## Non-Linear Models Cont'd: Infectious Diseases

Eradication & Control for the Models above:

- For SIR model, for eqn(5.37) $S_0 \approx N$, the total pop'n.
- So, from eqn(5.37) & eqns(5.40,5.37) for SIS & SIRS models, respectively *infectious contact number*,

$$\sigma \text{ or } R_0 = \beta N/\nu$$

  is important.
- It is population fraction in contact with an infective during infectious period.
- $\equiv$ mean number of secondary cases one infected case causes without interventions to control the infection.
- It helps model spread of an infectious disease thro a pop'n.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- From $\beta N/\nu$ can reduce infectious disease spread by:
  1. $\uparrow \nu$, removal rate of infectives.
  2. $\beta \downarrow$, infection rate btw $S$ & $I$. E.g. Disinfection, movement controls in Foot-and-Mouth.
  3. Reduce effective number $N$ which has the effect of $\downarrow S$.
  4. Again for the Foot-and-Mouth example:
     - Slaughter potential contacts surrounding infected farms was employed (politically controversial).
     - Vaccination of susceptibles for an epidemic.



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- Immunizing a population from a disease is impractical due to cost
- Logistics of administering vaccine to millions is difficult.
- Also costs:
  - Direct costs of producing & administering the dosage and
  - Indirect costs of public info campaigns
  - Indirect costs of bureaucracy to ensure all have been vaccinated.
- Thus would like to be able to provide safety from disease at the lowest possible cost.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- Actually only have to immunize a fraction of population to give *herd immunity*.
- Specifically must reduce *effective* value of $N$ so that disease dies out.
- From SIR model must move enough people such that (from eqn(5.31)), $\beta S_0 - \nu < 0$ so that $\dot{I} < 0$.
- i.e. need to lower epidemic threshold below one.
- Or fraction of people that must immunized is such that $S_0 < \nu/\beta$
- i.e. $100 \times (1 - \nu/\beta)$ percent of susceptibles must be immunized.



Notes

## Non-Linear Models Cont'd: Infectious Diseases

- This makes sense as if $\nu$ is small, it takes longer to recover from infection
- So an infective has more time to infect people.
- Thus as $\nu \downarrow, 1 - \nu/\beta \uparrow$; need to inoculate a larger fraction of the population.
- As $\beta \uparrow$, each infected person contacts more people in a given period and $1 - \nu/\beta \uparrow$.
- Thus again need to inoculate a larger fraction of population.



Notes



## Non-Linear Models Cont'd: Infectious Diseases

- $R_0$ & $1 - 1/R_0$ shown in % Table 5.2 for common diseases.

| Disease | Transmission | $R_0$ | $1 - \frac{1}{R_0}$ % |
|---|---|---|---|
| Measles | Airborne | 12 to 18 | 92 to 94.5 |
| Pertussis | Airborne droplet | 12 to 17 | 92 to 94 |
| Diphtheria | Saliva | 6 to 7 | 84 |
| Smallpox | Social contact | 5 to 7 | 80 to 85 |
| Polio | Fecal-oral route | 5 to 7 | 80 to 85 |
| Rubella | Airborne droplet | 5 to 7 | 80 to 85 |
| HIV/AIDS | Bodily Fluids | 2 to 5 | 50 to 80 |
| SARS | Airborne droplet | 2 to 5 | 50 to 80 |
| Influenza (1918) | Airborne droplet | 2 to 3 | 50 to 80 |
| Cholera | Fecal-oral route | 2.9 | 65.5 |

TABLE 5.2 : Values for $R_0$ for Several Common Infectious Diseases



Notes

## Non-Linear Models Cont'd: The Chemostat Revisited

The Chemostat Revisited

- Returning to chemostat above, can look at phase-plane plot. Derived the equations:

$$\frac{dn}{d\tau} = f(n,c) = \alpha_1\left(\frac{nc}{1+c}\right) - n \qquad (5.41)$$

and

$$\frac{dc}{d\tau} = g(n,c) = -\left(\frac{nc}{1+c}\right) - c + \alpha_2 \qquad (5.42)$$

containing (dimensionless) parameters:

$$\alpha_1 = \frac{VK_{max}}{F} \text{ and } \alpha_2 = \frac{C_0}{K_n}$$



Notes



## Non-Linear Models Cont'd: The Chemostat Revisited

- Eqns(5.41, 5.42) also contain dimensionless time, bacterial population & nutrient concentrations respectively:

$$\tau = \frac{tF}{V},\ n = \frac{N\alpha VK_{max}}{FK_n},\ c = \frac{C}{K_n}$$

- The phase-plane plot for $nVc$ is shown in Fig 5.10.
- The figure shows that when $\alpha_1 = 3, \alpha_2 = 1$ there are two equilibrium points:

$$(\bar{n}_1, \bar{c}_1) = \left(\alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1}\right) = (\frac{3}{2}, \frac{1}{2})$$

and

$$(\bar{n}_2, \bar{c}_2) = (0, \alpha_2) = (0, 1)$$

as predicted.



Notes

# Non-Linear Models Cont'd: The Chemostat Revisited

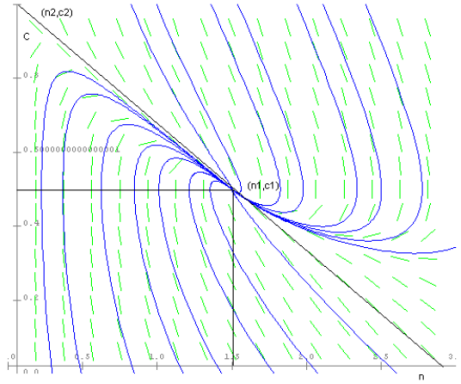



FIGURE 5.10 : Chemostat Phase-Plane Plot, $\alpha_1 = 3, \alpha_2 = 1$



Notes

Notes