Non-Linear Models

- Focus on more sophistiated interaction models between systems.
- These lead to *non-linear*, rather than linear, DEs; often not soluble exactly in analytical form so use *Phase-Plane Analysis*.
- This is a method where a system of DEs is reduced to a single DE, e.g.

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \tag{5.13}$$

  for some non-linear functions f, g, is written in terms of x, y only.
- Resulting curves plotted in (x, y) plane are known as *phase-plane trajectories*.

Notes

Non-Linear Models Cont'd

- This section considers three types of interaction models:
  1. Mutually destructive interaction,
  2. Interaction beneficial to one or the other species,
  3. Mutually beneficial interaction (symbiosis)
- Then we proceed to the modelling of infectious diseases.

Notes

Non-Linear Models Cont'd: Guerrilla Combat

The Guerrilla Combat Model

- Can model Combat btw occupying & guerrilla force (with some assumptions) as 2 coupled DEs.
- Combat models used to understand what factors affect battle outcomes e.g. how many occupying needed for victory.
- Let number of 'friendly' soldiers be x & 'enemy' (i.e. guerrilla forces) be y at time t.
- Assume
  - Large numbers permit use of continuous variables.
  - Major attrition factor is number of soldiers killed by opponents.
  - No prisoners taken & both use only gunfire (reasonable in guerrilla combat).

Notes

Non-Linear Models Cont'd: Guerrilla Combat

- If δx & δy are changes in armies by opponents' gunfire then

$$\delta x = -R_y P_y y \delta t \text{ and } \delta y = -R_x P_x x \delta t \qquad (5.14)$$

  This says that number of soldiers hit in a small time δt is equal to product of
  1. firing rate of each soldier (const. R_x, R_y for each army),
  2. probability single shots hit targets (P_x, P_y for each army, subscript is side firing),
  3. the number of soldiers firing.
- Firing rates R_x, R_y are assumed constant,

Notes

Non-Linear Models Cont'd: Guerrilla Combat

- P_x, P_y depend on whether target is exposed/ hidden.
- For 'friendly' troops, make P_y constant as each single shot has same hit probability.
- For hidden targets, P_x is ratio of area α exposed, to total area occupied by enemy soldiers, so $P_x = \alpha y/A$.
- Put into Eqn.(5.14) & letting $\delta x, \delta y, \delta t \to 0$:

$$\begin{aligned} \frac{dx}{dt} &= -ay \\ \frac{dy}{dt} &= -bxy \end{aligned} \qquad (5.15)$$

  where $a = R_y P_y$ and $b = R_x \alpha / A$ are positive constants.
- Eqn(5.15) are *Lanchester's Linear Law* for undirected fire.

Notes

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Non-Linear Models

Non-Linear Models Cont'd: Guerrilla Combat

- Eqns.(5.15) have equilibrium point (X, Y)

$$0 = -aY \text{ and } 0 = -bXY \qquad (5.16)$$

  i.e. Guerrilla army defeated with X unspecified.
- But phase-plane analysis shows some initial conditions give opposite outcomes.
- Taking Eqn.s(5.15) & dividing one by other get:

$$\frac{dy}{dx} = \frac{b}{a}x \qquad (5.17)$$

  This is first-order separable & may be integrated:

$$y = \frac{b}{2a}x^2 + K \qquad (5.18)$$

  where $K = y_0 - \frac{b}{2a}(x_0)^2$ is a constant.

Notes

Non-Linear Models Cont'd: Guerrilla Combat

- Fig 5.3 shows different parabolic solutions for different K.
- As $x, y > 0$ (+ive numbers), $\dot{x},\ \dot{y} < 0$ so curves go to $(0, 0)$.
  - 'Friendly' wins when $x > 0$ & $y = 0$, i.e. $K < 0$ & thus $y_0 < \frac{b}{2a}(x_0)^2$ at $t = 0$.
  - 'Away' wins when $x = 0$ & $y > 0$, so $K > 0$ & so $y_0 > \frac{b}{2a}(x_0)^2$.
  - All dead when
$$K = 0 \text{ \& so } \frac{y_0}{(x_0)^2} = \frac{b}{2a}.$$
- Guerrillas have advantage as $b << a$ cos ratio of areas $\alpha/A << 1$,
- So two armies are evenly matched if x_0 is large and y_0 is relatively small.

Notes

Non-Linear Models Cont'd: Guerrilla Combat

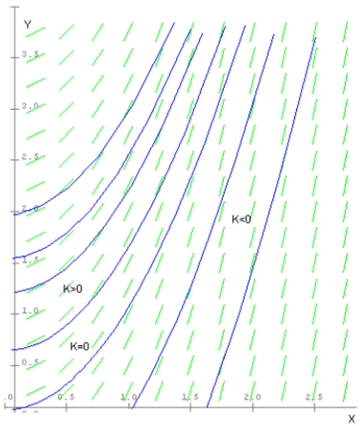

FIGURE 5.3 : Guerrilla Combat Phase-Plane Plot

Notes

Non-Linear Models Cont'd: Predator-Prey

The Predator-Prey Model

- Guerrilla Combat is mutually destructive (i.e. any interaction results in pop'n decrease).
- More usual for pop'n increase of one species ↓ & other ↑, - *predator-prey*.
- First models by Lotka (1925) & Volterra (1926) for fish in Mediterranean.
- Lotka-Volterra model is based on a number of assumptions:
  1. Prey grow in an unlimited way when no predation.
  2. Predators depend on prey to thrive/ survive.
  3. Predation rate depends on likelihood that a predator encounters a victim.
  4. Predator pop'n growth rate $\propto$ rate of predation.

Notes

Non-Linear Models Cont'd: Predator-Prey

- These can be expressed mathematically as:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned} \qquad (5.19)$$

  where x, y are prey & predator numbers & a, b, c, d are positive constants.
- in Eqn.(5.19a), item 1 is the ax term & item 2 the $-bxy$ term.
- in Eqn.(5.19b), dxy term corresponds to item 3 & $-cy$ term corresponds to item 4.

Notes

Non-Linear Models Cont'd: Predator-Prey

- Note: Eqn.s(5.19) depend on 4 parameters a, b, c, d
- Removing dimensions by putting $x = \frac{c}{d}u$, $y = \frac{a}{b}v$, & $t = \frac{1}{a}\tau$, Eqn.s(5.19) become:

$$\begin{aligned} \frac{du}{d\tau} &= u(1-v) \\ \frac{dv}{d\tau} &= \gamma v(u-1) \end{aligned} \quad (5.20)$$

where $\gamma = \frac{c}{a}$.

- In the u, v phase plane, these give:

$$\frac{dv}{du} = \gamma \frac{v(u-1)}{u(1-v)} \quad (5.21)$$

with equilibrium points at $(\bar{u}, \bar{v}) = (0,0)$ & $(\bar{u}, \bar{v}) = (1,1)$.

Notes

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Non-Linear Models Cont'd: Predator-Prey

- Integrate this directly to give solution for the trajectories:

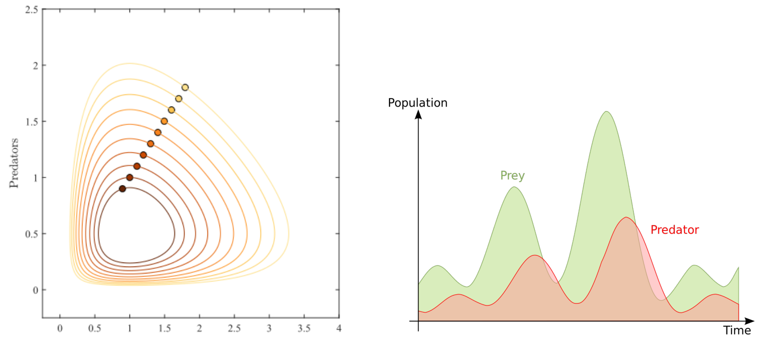
$$\gamma u + v - \ln u^{\gamma} v = H \quad (5.22)$$

for some $H > H_{min} = 1 + \gamma$.

- H_{min} is minimum of H over all (u, v) & it occurs at $u = v = 1$.
- For a given $H > H_{min}$, trajectories given by Eqn.(5.22) are closed & given in Fig. 5.4(a).

Notes

Non-Linear Models Cont'd: Predator-Prey



(a) Phase-Plane Plot (b) Time Series Plot

FIGURE 5.4 : Predator Prey Plots for $\gamma = 2$

Notes

Non-Linear Models Cont'd: Predator-Prey

- As noted above, system's equilibrium points are $(\bar{u}, \bar{v}) = (0,0)$ & $(\bar{u}, \bar{v}) = (1,1)$.
- As in the chemostat, write Jacobian $\mathbf{A}(u,v)$ of Eqn.s(5.20) as:

$$\mathbf{A}(u,v) = \begin{pmatrix} 1-v & -u \\ \gamma v & \gamma(u-1) \end{pmatrix} \quad (5.23)$$

- At $(\bar{u}, \bar{v}) = (0,0)$ this reduces to diagonal matrix with eigenvalues $\lambda_1 = 1$ & $\lambda_2 = -\gamma$.
- Hence, solutions *near* $(0,0)$ are

$$\mathbf{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}.$$

where $\mathbf{x}_1, \mathbf{x}_2$ are e-vectors of $\mathbf{A}$ at $(0,0)$, as per Eqn.3.20

Notes

Non-Linear Models Cont'd: Predator-Prey

- Since $-\gamma < 0 < 1$, say that $(0, 0)$ is a *saddle point* as solution

$$\mathbf{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = c_1 \mathbf{x}_1 e^t + c_2 \mathbf{x}_2 e^{-\gamma t}. \quad (5.24)$$

  has eigenvalues of different signs & so is like a saddle.
- Shows exp'l growth & decay of prey & predator respectively on the u, v axes referred to in items 1 & 2 above.
- As one $\lambda_i > 0$ always, rhs of Eqn.5.24 increases exp'ly with t so $(0, 0)$ is unstable equilibrium point.

Notes

Non-Linear Models Cont'd: Predator-Prey

- At the equilibrium $(\bar{u}, \bar{v}) = (1, 1)$, the Jacobian is:

$$\mathbf{A}(1, 1) = \begin{pmatrix} 0 & -1 \\ \gamma & 0 \end{pmatrix} \quad (5.25)$$

  e-values of which are imaginary $\pm i\sqrt{\gamma}$, so point is a *centre*.
- Mathematically complex eigenvalues are only stable if in left half of Argand Diagram.
- Since these lie *on* the axis, model is *unstable* mathematically.
- The reason for this can be seen qualitatively from Fig 5.4(a).
- Solutions are periodic, oscillating around a fixed point in the phase-plane of (u, v), Fig 5.4(b)).

Notes

Non-Linear Models Cont'd: Predator-Prey

- Different trajectories correspond to different initial conditions in Fig 5.4(a)
- These are closed curves for different values of H in Eqn.(5.22).
- Viewing areas round the centre $(1, 1)$ as quadrants, look at slopes given by Eqn.s(5.20) & hence trajectories' directions.
  - If 1st quadrant is closest to the origin, Eqn.s(5.20) shows $\dot{u} > 0, \dot{v} < 0$
  - Shows that prey increase & predators decrease.
  - So phase-plane trajectories move in a counter-clockwise direction.

Notes

Non-Linear Models Cont'd: Predator-Prey

- Although attempts have been made using LV model in real-world oscillatory phenomena, instability issue remains.
- However model allows conclusions to be drawn on qualitative behaviour of such a system
- Using purely mathematical methods and very simple assumptions about the system.
- Most important is that systems like Eqn.s(5.19) give rise to the existence of periodic solutions.

Notes

Non-Linear Models Cont'd: Symbiosis

Symbiosis

- 3rd interaction model is *symbiosis* where interaction benefits all species.
- Model has form:

$$\begin{aligned} \frac{dx}{dt} &= \mu_1 x \left(1 - \frac{x}{K_1} + c_{12}\frac{y}{K_1}\right) \\ \frac{dy}{dt} &= \mu_2 y \left(1 - \frac{y}{K_2} + c_{21}\frac{x}{K_2}\right) \end{aligned} \quad (5.26)$$

- x, y grow logistically in absence each other with different carrying capacities K_1, K_2 respectively.
- Positive parameters c_{12}, c_{21} show positive effect that each species has on the other.

Notes

Non-Linear Models Cont'd: Symbiosis

- Setting $u_1 = \frac{x}{K_i}$, $u_2 = \frac{y}{K_2}$, & $\tau = \mu_1 t$, Eqn.(5.26) goes to:

$$\begin{aligned} \frac{du_1}{d\tau} &= u_1 \left(1 - u_1 + \alpha_{12} u_2\right) \\ \frac{du_2}{d\tau} &= \xi u_2 \left(1 - u_2 + \alpha_{21} u_1\right) \end{aligned} \quad (5.27)$$

  where (non-dim'l) parameters given by $\xi = \mu_2/\mu_1$, $\alpha_{12} = c_{12}K_2/K_1$ & $\alpha_{21} = c_{21}K_1/K_2$.
- The steady-states of Eqn.(5.27) are given by: $(\dot{u}_1, \dot{u}_2) = (0, 0) \equiv (u, v) = (0, 0)$ or $(1 + \alpha_{12}u_2, 1 + \alpha_{21}u_1)$
- Leads to 4 cases:

$$(0,0) \text{ or } (0,1) \text{ or } (1,0) \text{ or } \left(\frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}}, \frac{1+\alpha_{21}}{1-\alpha_{12}\alpha_{21}}\right)$$

  where final case is only relevant if $\alpha_{12}\alpha_{21} < 1$.

Notes

Non-Linear Models Cont'd: Symbiosis

- Next, can find Jacobian matrix, at a steady-state $(\bar{u}_1, \bar{u}_2)$:

$$\mathbf{A}(\bar{u}_1, \bar{u}_2) = \begin{pmatrix} 1 - 2\bar{u}_1 + \alpha_{12}\bar{u}_2 & \alpha_{12}\bar{u}_1 \\ \xi\alpha_{21}\bar{u}_2 & \xi(1 - 2\bar{u}_2 + \alpha_{21}\bar{u}_2) \end{pmatrix} \quad (5.28)$$

- So at the equilibrium points, get

$$\text{for } (0,0) \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$$

$$\text{for } (1,0) \quad \mathbf{A} = \begin{pmatrix} -1 & \alpha_{12} \\ 0 & \xi(1+\alpha_{21}) \end{pmatrix}$$

$$\text{for } (0,1) \quad \mathbf{A} = \begin{pmatrix} 1+\alpha_{12} & 0 \\ \xi\alpha_{21} & -\xi \end{pmatrix}$$

Notes

Non-Linear Models Cont'd: Symbiosis

$$\text{for } \left(\frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}}, \frac{1+\alpha_{21}}{1-\alpha_{12}\alpha_{21}}\right),$$

$$\mathbf{A} = \frac{1}{1-\alpha_{12}\alpha_{21}} \begin{pmatrix} -(1+\alpha_{12}) & \alpha_{12}(1+\alpha_{12}) \\ \xi\alpha_{21}(1+\alpha_{21}) & -\xi(1+\alpha_{21}) \end{pmatrix}$$

- α's +ive, $\Rightarrow$ $(0,0)$ is unstable node, $(1,0)$ & $(0,1)$ saddles.
- 4th steady-state has e-values given by zeros of characteristic polynomial of Jacobian, hence

$$\lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0 \quad (5.29)$$

Notes

Non-Linear Models Cont'd: Symbiosis

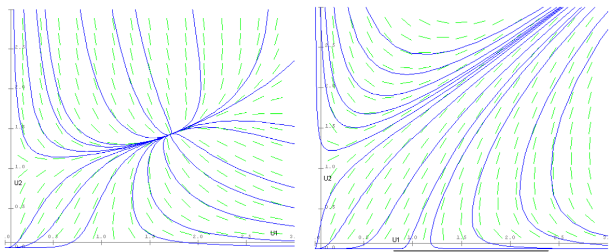
- Stability requires $\Re(\lambda_1, \lambda_2)$ being -ive, only true if coeffs of eqn(5.29) +ive
- Thus need trace$(\mathbf{A}) < 0$ - always true for $\alpha_{12}\alpha_{21} < 1$ and

$$\det(\mathbf{A}) > 0 \equiv \frac{\xi(1+\alpha_{12})(1+\alpha_{21})\left[1+\alpha_{12}\alpha_{21}\right]}{1-\alpha_{12}\alpha_{21}} > 0$$

  which is again satisfied with $\alpha_{12}\alpha_{21} < 1$.
- See this on phase-plane plots in Fig 5.5 (a),(b).
- Here $\alpha_{12} = 0.5,\ \alpha_{21} = 0.25\ \&\ \xi = 2$ in Fig 5.5(a) (i.e. $\alpha_{12}\alpha_{21} < 1$)
- Gives a stable equilibrium (i.e. stable node) for the fourth case at $\left(\frac{12}{7}, \frac{10}{7}\right)$.
- Fig 5.5(b) shows a similar plot with $\alpha_{12} = 1.5,\ \alpha_{21} = 1.25\ \&\ \xi = 2$ (i.e. $\alpha_{12}\alpha_{21} > 1$) giving instabilities everywhere.

Notes

Non-Linear Models Cont'd: Symbiosis



(a) (u_1, u_2) for $\alpha_{12}\alpha_{21} < 1$ (b) (u_1, u_2) for $\alpha_{12}\alpha_{21} > 1$

FIGURE 5.5 : Symbiosis Phase-Plane Plots

Notes

Non-Linear Models Cont'd: The Chemostat Revisited

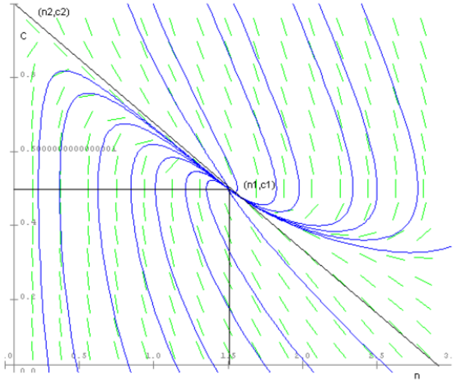


FIGURE 5.11 : Chemostat Phase-Plane Plot, $\alpha_1 = 3, \alpha_2 = 1$

Notes

Notes