Applications of Non-Linear Models: Logistic Growth

- Linear difference equations are useful as permit closed-form solutions to be easily obtained.
- However, solutions often have don't agree with observation.
- In many areas of biology & esp. population biology, non-linear models are better.
- Section looks at some simple non-linear models for population growth over time.
- The simplest model is the *logistic equation*
- Introduce the logistic equation in discrete & later continuous form.

Notes

Applications of Non-Linear Models: Logistic Growth cont'd

- Can't use linear model $M_{n+1} = aM_n$ for real populations due to unbounded growth with n.
- So if express it in the form:

$$M_{n+1} = M_n + r \times M_n \tag{3.56}$$

$r \equiv$ no. of births $-$ no. of deaths per time period. & replace r by $R(M_n)$ (non-constant growth rate) get *Logistic Growth Eqn*:

$$M_{n+1} = M_n + kM_n\left(1 - \frac{M_n}{K}\right). \tag{3.57}$$

where

$$R(M_n) = k - \frac{k}{K}M_n,$$

for growth rate k & population *carrying capacity* K (limits growth rate k).

Notes

Applications of Non-Linear Models: Logistic Growth cont'd

- In Logistic Growth Eqn (3.57)

$$M_{n+1} = M_n + kM_n\left(1 - \frac{M_n}{K}\right)$$

term in brackets behaves as follows:
  - For small M_n, $\frac{M_n}{K}$ is small & growth is unbounded.
  - For large M_n, $\frac{M_n}{K} \to 1$ & Eqn (3.57) behaves like

$$M_{n+1} = M_n + \epsilon$$

  for small ϵ.
- So for large M_n overcrowding slows growth rate to zero.

Notes

Applications of Non-Linear Models: Logistic Growth cont'd

- By writing Eqn.(3.57) in the form:

$$M_{n+1} - M_n = kM_n\left(1 - \frac{M_n}{K}\right). \qquad (3.58)$$

notice some thing; as LHS of eqn(3.58) is population *change* between successive times...
  - if $M_n < K$ then population in next time interval increases & decrease for $M_n > K$.
  - expect steady increase for small M_n but small oscillations above & below *K* for large populations.
- Model is realistic here mirroring real populations.
- Numerical problems (i.e. -ve M_n) happen for certain k in Eqn.(3.58) & reduce applicability of model.

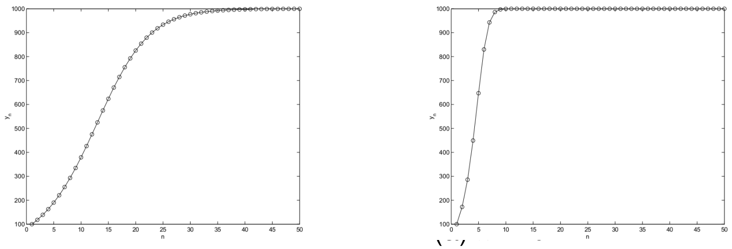
Notes

Applications of Non-Linear Models: Logistic Growth cont'd

- To see how Logistic Growth model performs, look at plots of M_n V n for various k.
- k is average fertility of an individual in the population.
- k varies $0 < k \leq 3$ for fixed carrying capacity $K = 1000$ & initial population $M_0 = 100$.

Notes

Applications of Non-Linear Models: Logistic Growth cont'd



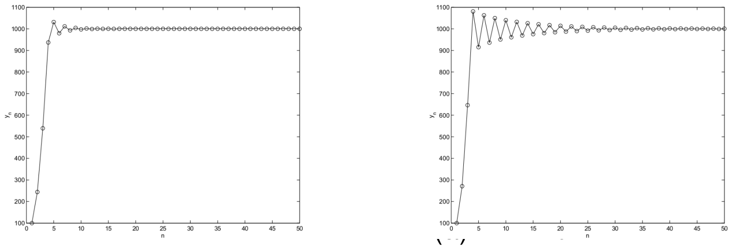
(b) $k = 0.8$

FIGURE 3.7 : Logistic Equation: Stable Growth

- Populations in Fig 3.7 (a),(b) approach K as n increases.
- Growth initially exp'l but K causes pop'n to level out.
- Larger k causes steeper growth, more rapid overcrowding.

Notes

Applications of Non-Linear Models: Logistic Growth cont'd



(b) $k = 1.9$

FIGURE 3.8 : Logistic Equation: Damped Oscillations

- Early on, pop'n growth is rapid &overshoots K before overcrowding is felt.
- Note: as k increases over 1, takes longer for oscillations to die out.

Notes

Applications of Non-Linear Models: Logistic Growth cont'd

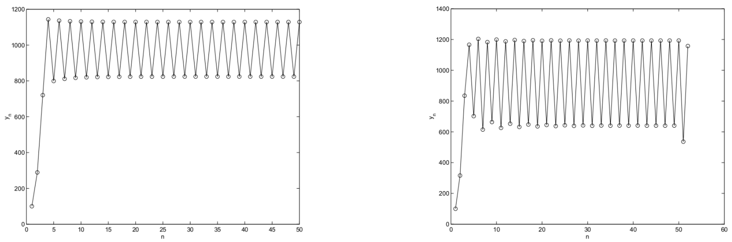


FIGURE 3.9 : Logistic Equation: Cyclic Growth

- In Fig 3.9(a,b) plot populations with $2 < k < 2.57$.
- Pop'n not damped but fluctuates above & below K, comes back every 2nd breeding season, a *2-cycle*.

Notes

Applications of Non-Linear Models: Logistic Growth cont'd

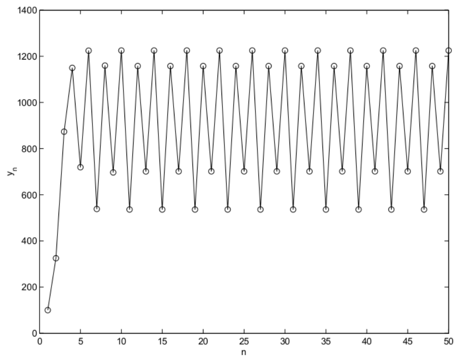


FIGURE 3.10 : Logistic Equation: $k = 2.5$, a *4-cycle*

- As $k \to 2.57$, pop'n repeats every 4th breeding season, a *4-cycle*.

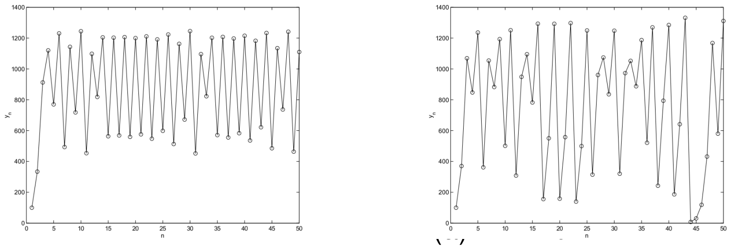
Notes

Applications of Non-Linear Models: Logistic Growth cont'd

- Beyond $k = 2.5$ (& for $k < 2.57$), *4-cycles* become *8-cycles* become *16-cycles* - *period doubling*.
- For $2.57 < k < 3$ (Fig. 3.11 a,b), though behaviour still predicted by difference eqn, growth pattern *seems* more random for increasing k.
- This *chaotic* behaviour was important in modelling as questioned fact that external events *always* caused complex population fluctuations.
- Often hard to say if they are chaotic or simply long-term periodic.
- Feature of chaos is sensitivity to initial population: if M_0 differs, can cause big pop'n changes

Notes

Applications of Non-Linear Models: Logistic Growth cont'd



(b) $k = 3.0$

FIGURE 3.11 : Logistic Equation: Into the Chaotic

Notes

Applications of Non-Linear Models: Logistic Growth cont'd

- Why does pop'n oscillate at all? Answer is twofold:
  - pop'n is *self-regulating* through pop'n-dependent growth rate,
  - regulating effect felt in *next* time interval but *determined* in *current*.
- Gives rise to natural delay as pop'n responds to overcrowding & a corresponding over-compensation when growth rate is sufficiently high.
- Over-compensation can lead to oscillatory behaviour & chaos.
- Nb: if large pop'n (or small timestep) that continuous breeding is assumed, growth rate responds instantaneously to pop'n & oscillations do not normally occur.
- Differential equations will be used to show this below.

Notes