A Small Wager (cont'd)

Which may be shown to be:

$$\mathbf{u}_n = \begin{pmatrix} 1.7 & -0.65 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.7^n & 0 \\ 0 & (-0.65)^n \end{pmatrix} \times \begin{pmatrix} 0.43 & 0.28 \\ -0.43 & 0.78 \end{pmatrix} \mathbf{u}_0$$

Which reduces to

$$\mathbf{u}_n = 0.43 \begin{bmatrix} (1.7^{n+1}) - (-0.65)^{n+1} \\ 1.7^n - (-0.65)^n \end{bmatrix} \quad (3.41)$$

for an initial sum of $\mathbf{u}_0 = (1, 0)^T$, this gives$\mathbf{u}_1 \approx \begin{pmatrix} 1.05 \\ 1 \end{pmatrix}$ etc.

Notes

The Leslie Matrix

- The *Leslie* matrix is a generalization of the above.
- It describes annual increases in various age categories of a population.
- As above we write $\mathbf{p}_{n+1} = \mathbf{A}\mathbf{p}_n$ where $\mathbf{p}_n$, $\mathbf{A}$ are given by:

$$\mathbf{p}_n = \begin{pmatrix} p_n^1 \\ p_n^2 \\ \vdots \\ p_n^m \end{pmatrix}, \; \mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{m-1} & \alpha_m \\ \sigma_1 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \sigma_{m-1} & 0 \end{pmatrix} \quad (3.42)$$

α_i, σ_i, the number of births in age class i in year n & probability that i year-olds survive to $i + 1$ years old, respectively.

Notes

The Leslie Matrix cont'd

- Long-term population demographics found as with Eqn.(3.21) using λ_is of $\mathbf{A}$ in Eqn.(3.42) & det$(\mathbf{A} - \lambda\mathbf{I}) = 0$ to give Leslie characteristic equation:

$$\lambda^n - \alpha_1\lambda^{n-1} - \alpha_2\sigma_1\lambda^{n-2} - \alpha_3\sigma_1\sigma_2\lambda^{n-3} - \cdots - \alpha_n\prod_{i=1}^{n-1}\sigma_i = 0 \quad (3.43)$$

α_i, σ_i, are births in age class i in year n & the fraction that i year-olds live to $i+1$ years old, respectively.

Notes

The Leslie Matrix cont'd

- Eqn.(3.43) has one $+$ive eigenvalue λ^* & corresponding eigenvector, $\mathbf{v}^*$.
- For a general solution like Eqn.(3.19)

$$\mathbf{P}_n = c_1\lambda_1^n\mathbf{v}_1 + \cancelto{0}{c_2\lambda_2^n\mathbf{v}_2} + \cdots + \cancelto{0}{c_n\lambda_m^n\mathbf{v}_m},$$

with dominant eigenvalue $\lambda_1 = \lambda^*$ gives long-term solution:

$$\mathbf{P}_n \approx c_1\lambda_1^n\mathbf{v}_1 \quad (3.44)$$

with *stable age distribution* $\mathbf{v}_1 = \mathbf{v}^*$. The *relative* magnitudes of its elements give stable state proportions.

Notes

The Leslie Matrix cont'd

Example 3: Leslie Matrix for a Salmon Population

- Salmon have 3 age classes & females in the 2nd & 3rd produce 4 & 3 offspring, each season.
- Suppose 50% of females in 1st age class survive to 2nd age class & 25% of females in 2nd age class live on into 3rd.
- The Leslie Matrix (c.f. Eqn.(eqn:1.14)) for this population is:

$$\mathbf{A} = \begin{pmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{pmatrix} \tag{3.45}$$

- Fig. 3.4 shows the growth of age classes in the population.

Notes

Leslie Matrix cont'd

Example 3: Leslie Matrix for a Salmon Population

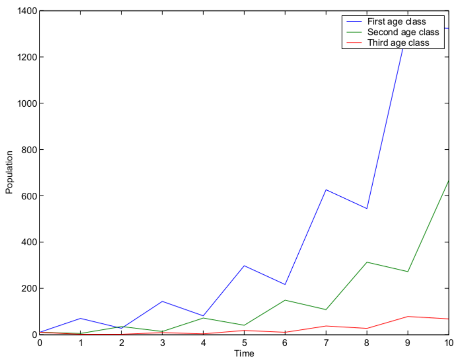

FIGURE 3.4 : Growth of Salmon Age Classes

Notes

The Leslie Matrix cont'd

Example 3: Leslie Matrix for a Salmon Population

- The eigenvalues of the Leslie matrix may be shown to be

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & -1.309 & 0 \\ 0 & 0 & -0.191 \end{pmatrix} \quad (3.46)$$

and the eigenvector matrix $\mathbf{S}$ to be given by

$$\mathbf{S} = \begin{pmatrix} 0.9474 & 0.9320 & 0.2259 \\ 0.3158 & -0.356 & -0.591 \\ 0.0526 & 0.0680 & 0.7741 \end{pmatrix} \quad (3.47)$$

- Dominant e-vector: $(0.9474, 0.3158, 0.0526)^T$, can be *normalized* (divide by sum), to $(0.72, 0.24, 0.04)^T$.

Notes

The Leslie Matrix cont'd

Example 3: Leslie Matrix for a Salmon Population cont'd

- Long-term, 72% of pop'n are in 1st age class, 24% in 2nd and 4% in 3rd.
- Thus, due to principal e-value $\lambda_1 = 1.5$, population increases.
- Can verify by taking *any* initial age distribution & multiplying it by **A**.
- It always converges to the proportions above.

Notes

The Leslie Matrix cont'd

A side note on matrices similar to the Leslie matrix.
Any *lower diagonal* matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.48}$$

Can 'move' a vector of age classes forward by 1 generation e.g.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \tag{3.49}$$

Notes

Stability in Difference Equations

- If difference equation system has the form $\mathbf{u}_n = \mathbf{A}\mathbf{u}_{n-1}$ then growth as $n \to \infty$ depends on the λ_i thus:
  - If *all* eigenvalues $|\lambda_i| < 1$, system is *stable* & $\mathbf{u}_n \to 0$ as $n \to \infty$.
  - Whenever *all* values satisfy $|\lambda_i| \leq 1$, system is *neutrally stable* & $\mathbf{u}_n$ is bounded as $n \to \infty$.
  - Whenever *at least* one value satisfies $|\lambda_i| > 1$, system is *unstable* & $\mathbf{u}_n$ is unbounded as $n \to \infty$.

Notes

Markov Processes

- Often with difference equations don't have *certainties* of events, but *probabilities*.
- So with Leslie Matrix Eqn.(3.42):

$$\mathbf{p}_n = \begin{pmatrix} p_n^1 \\ p_n^2 \\ \vdots \\ p_n^m \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{m-1} & \alpha_m \\ \sigma_1 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_{m-1} & 0 \end{pmatrix} \quad (3.50)$$

  σ_i is probability that i year-olds survive to $i+1$ years old.
- Leslie model resembles a *discrete-time Markov chain*
  - Markov chain: discrete random process with *Markov property*
  - Markov property: state at t_{n+1} depends only on that at t_n.
- The difference between Leslie model & Markov model, is:
  - In Markov $\alpha_m + \sigma_m$ must $= 1$ for each m.
  - Leslie model may have these sums $<> 1$.

Notes

Markov Processes cont'd

Stochastic Processes

- A Markov Process is a particular case of a *Stochastic Process*.
- *Stochastic Process* is where probabilities govern entering a state.
- A Markov Process is a Stochastic Process where probability to enter a state depends only on the last state occupied
- as well as on the *Transition* matrix governing the process.
- If Transition Matrix terms are constant from one timestep to the next, process is *Stationary*.

Notes

Markov Processes cont'd

- General form of *discrete-time Markov chain* is given by:

$$\mathbf{u}_{n+1} = \mathbf{M}\mathbf{u}_n$$

where $\mathbf{u}_n$, $\mathbf{M}$ are given by:

$$\mathbf{u}_n = \begin{pmatrix} u_n^1 \\ u_n^2 \\ \vdots \\ u_n^p \end{pmatrix}, \ \mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1\ p-1} & m_{1p} \\ m_{21} & m_{22} & \dots & m_{2\ p-1} & m_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{p1} & m_{p2} & \dots & m_{p\ p-1} & m_{pp} \end{pmatrix} \quad (3.51)$$

- **M** is $p \times p$ *Transition* matrix & its m_{ij} terms are called *Transition probabilities* such that $\sum_{i=1}^{p} m_{ij} = 1$.
- m_{ij} is probability that that item goes from state i at t_n to state j at t_{n+1}.

Notes

Markov Processes cont'd

Example 4: Two Tree Forest Ecosystem

- In a forest there are only two kinds of trees: oaks and cedars.
- At any time n sample space of possible outcomes is (O, C)
- Here O $=$ % of tree *population* that is oak in a particular year and C, $=$ % that is cedar.
- If same life spans & on death same chance an oak is replaced by an oak or a cedar
- But that cedars are more likely ($p = 0.74$) to be replaced by an oak than another cedar ($p = 0.26$).
- How can we track changes in the different tree types with time?

Notes

Markov Processes cont'd

Example 4: Two Tree Forest Ecosystem

- This is a Markov Process as oak/cedar fractions at t_{n+1} etc are defined by those at t_n.
- Transition Matrix (from Eqn.(3.51)) is Table 3.1:

		From	
		Oak	Cedar
To	Oak	0.5	0.74
	Cedar	0.5	0.26

TABLE 3.1 : Tree Transition Matrix

- Table 3.1 in matrix form is:

$$\mathbf{M} = \begin{pmatrix} 0.5 & 0.74 \\ 0.5 & 0.26 \end{pmatrix} \qquad (3.52)$$

Notes

Markov Processes cont'd

Example 4: Two Tree Forest Ecosystem

- To track system changes, let $\mathbf{u}_n = (o_n, c_n)^T$ be probability of oak & cedar after n generations.
- If forest is initially 50% oak and 50% cedar, then $\mathbf{u}_0 = (0.5, 0.5)^T$. Hence

$$\mathbf{u}_n = \mathbf{M}\mathbf{u}_{n-1} = \mathbf{M}^n\mathbf{u}_0 \qquad (3.53)$$

- **M** can be shown to have one positive λ & corresponding eigenvector $(0.597, 0.403)^T$
- This is the distribution of oaks and cedars in the nth generation.

Notes

Markov Processes cont'd

Example 5: Soft Drink Market Share

- In a soft drinks market there are two Brands: Coke & Pepsi.
- At any time n sample space of possible outcomes is (P, C)
- Here P $=$ % market share that is Pepsi's in one year and C, $=$ % that is Coke's.
- Know that chance of switching from Coke to Pepsi is 0.1
- And the chances of someone switching from Pepsi to Coke are 0.3.
- How can the changes in the different proportions be modelled?

Notes

Markov Processes cont'd

Example 5: Soft Drink Market Share

- This is a Markov Process as shares of Coke/Pepsi at t_{n+1} are defined by those at t_n.
- Transition Matrix (from Eqn.(3.51)) is Table 3.2:

		From	
		Coke	Pepsi
To	Coke	0.9	0.3
	Pepsi	0.1	0.7

TABLE 3.2 : Soft Drink Market Share Matrix

- Table 3.2 in matrix form:

$$\mathbf{M} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \qquad (3.54)$$

Notes

Markov Processes cont'd

Example 5 can also be represented using a *Transition Diagram*, thus:

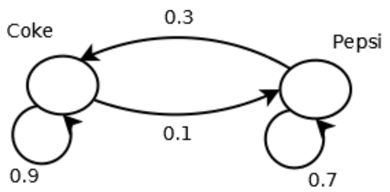

FIGURE 3.5 : Market Share Transition Diagram

- The eigenvalues of the matrix in Eqn(3.53) are $1, \frac{3}{5}$.
- The largest eigenvector, can be found to be $(0.75, 0.25)^T$.
- This is the proportions of Coke and Pepsi in the nth generation.

Notes

Markov Processes cont'd

Absorbing States

A state of a Markov Process is said to be *absorbing* or *Trapping* if $\mathbf{M}_{ii} = 1$ and $\mathbf{M}_{ij} = 0 \; \forall j$

Absorbing Markov Chain

A Markov Chain is *absorbing* if it has one or more absorbing states. If it has one absorbing state (for instance state i, then the steady state is given by the eigenvector **X** where $\mathbf{X}_i = 1$ and $\mathbf{X}_j = 0 \; \forall j \neq i$

Notes

Markov Processes cont'd

Example 5: Soft Drink Market Share, Revisited

- As Soft Drinks market is 'liquid', KulKola decides to trial product Brand 'X'.
- Despite its name, Brand 'X' has potential[1] to 'Shift the Paradigm' in Cola consumption.
- They think, inside 5 years, they can capture nearly all the market.
- Investigate if this is true, given that they take 20% of Coke's share and 30% of Pepsi's per annum.

[1] from KulKola's Marketing viewpoint

Notes

Markov Processes cont'd

Example 5: Soft Drink Market Share

- Again, shares of Coke/Pepsi/Brand 'X' at $n+1$ etc are defined by those at n.
- Transition Matrix (from Eqn.(3.51)) is Table 3.3:

		From		
		Coke	Pepsi	Brand 'X'
	Coke	0.6	0.4	0
To	Pepsi	0.2	0.3	0
	Brand 'X'	0.2	0.3	1

TABLE 3.3 : Soft Drink Market Share Matrix Revisited

- Table 3.3 in matrix form:

$$\mathbf{M} = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.2 & 0.3 & 0 \\ 0.2 & 0.3 & 1 \end{pmatrix} \quad (3.55)$$

Notes

Markov Processes cont'd

The Transition Diagram corresponding to Example 5 Revisited is:

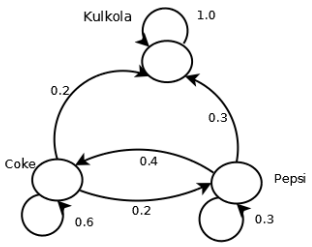

FIGURE 3.6 : Market Share Transition Diagram

- λ_{max} of the matrix in Eqn(3.55) is 1.
- $\mathbf{v}_{max}$, is $(0, 0, 1)^T$ giving the shares of Coke, Pepsi and Brand 'X' in the nth generation, respectively.

Notes

Markov Processes cont'd

- Markov Models have a visible state
- So the state transition probabilities & transition matrix can be noted by the observer.
- In another type of model, *Hidden Markov Models* this visibility restriction is relaxed
- The transition probabilities are generally not known.
- These kind of models are very useful in AI as well as many other applications.

Notes