Chapter 2:

Time Series Modelling

Notes

Chapter 3:

Discrete Models

Notes

Glossary of Terms

Here are some of the types of symbols you will see in this section:

Name	Symbol Face	Examples
Vector	Lower-case bold	$\mathbf{u}, \mathbf{v}, \mathbf{p}$
Matrix	Upper-case bold	$\mathbf{M}, \mathbf{X}, \mathbf{A}$
Vector at Time step	Subscript	$\mathbf{u_0}$
Age Category at Time step	Subscript & Superscript	u_0^t

Notes

First Order Linear Difference Equations

- We start with the most basic equations.
- State at time t purely related to that at $t-1$
- Example in nature is cell division

$$M_{n+1} = aM_n \tag{3.1}$$

a constant, n is the generation number

- So number in nth generation related to that in first by:

$$M_n = aM_{n-1} = \ldots = a^n M_0 \tag{3.2}$$

- So if
  1. $|a| > 1$ the population will increase,
  2. $|a| = 1$ the population will be stable,
  3. $|a| < 1$ the population will decrease.

Notes

Examples of Linear Difference Equations

Example 1: Rabbit Reproduction

- Difference eqn *Order* is how many terms determine present state.
- Examples of higher order difference eqns common in nature.
- Leonardo of Pisa (a.k.a.*Fibonacci*) modelled rabbit reproduction.
- Assumptions of Fibonacci model:
  1. Each pair of rabbits can reproduce from two months old
  2. Each reproduction produces only one pair of rabbits
  3. All rabbits survive.
- Number of rabbit pairs at t_{n+1}, M_{n+1} ($n =$months) is:

$$M_{n+1} = M_n + M_{n-1}. \qquad (3.3)$$

- With $M_0 = 1$, $M_1 = 1$, (1 pair at $t = 0$) number grows as $1, 1, 2, 3, 5, 8, 13, \ldots$

Notes

Examples of Linear Difference Equations Cont'd

Example 1: Rabbit Reproduction (cont'd)

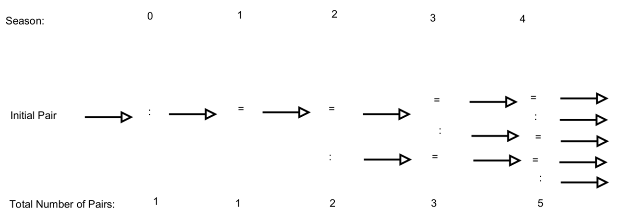

FIGURE 3.1 : Fibonacci Number of Immature (:) & Mature Rabbits (=)

Notes

Examples of Linear Difference Equations Cont'd

Example 1: Rabbit Reproduction (cont'd)

- Instead of Eqn.(3.3), 'one step' eqn (eg. Eqn.(3.1)) is better
- Get this by writing Eqn.(3.3) in the form:

$$\begin{aligned} M_{n+1} + M_n &= M_{n+2} \\ M_{n+1} &= M_{n+1} \end{aligned} \qquad (3.4)$$

which, by writing

$$\mathbf{u}_n = \begin{pmatrix} M_{n+1} \\ M_n \end{pmatrix}$$

takes the form

$$\mathbf{u}_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u}_n. \qquad (3.5)$$

Notes

Digression: Matrix Basics

Matrices & Vectors

- A matrix is an array of coefficients of the form:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad (3.6)$$

- This is an $m \times n$ matrix with m rows & n columns.
- A vector is an array of coefficients of the form:

$$\mathbf{A} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

Notes

Digression: Matrix Basics cont'd

Matrix Systems

- In the course will see systems of equations of the form:

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array} \quad (3.7)$$

  for a system of two equations in 2 unknowns x_1, x_2constant coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ & right-hand side b_1, b_2.
- With matrix multiplication, this can be written as:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (3.8)$$

Notes

Digression: Matrix Basics cont'd

Matrix Inverse, Identity Matrix

- Can show that Eqn.(3.8) has a unique solution if $\mathbf{A}^{-1}$ exists.
- $\mathbf{A}^{-1}$ has the property:

$$\mathbf{A} \times \mathbf{A}^{-1} = \text{Identity Matrix } \mathbf{I}$$

- The $n \times n$ identity matrix is given by:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \quad (3.9)$$

Notes

Digression: Matrix Basics cont'd

Solutions to Matrix Systems: Matrix Determinant

- To solve $\mathbf{x} = (x, y)$ in Eqn.(3.8) must find $\mathbf{A}^{-1}$
- For a 2×2 Matrix $\mathbf{A}^{-1}$ is given by:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \quad (3.10)$$

where $\det(\mathbf{A})$ is the *determinant* of the matrix $\mathbf{A}$

- *determinant* of **A** is $\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$.
- Eqn.(3.10) holds for a 2×2 matrix only.
- The solution in Eqn.(3.8)) only exists if this condition is met:

$$\det(\mathbf{A}) \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \quad (3.11)$$

Notes

Intro to the Topic
Time Series
Discrete Models
Growth and Decay
Linear Difference Equations
Non-Linear Models with Applications

Digression: Matrix Basics cont'd

Matrix Characteristic Equation, Trace

- The *characteristic equation* for $\mathbf{A}$ is given by $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
- It arises in a number of circumstances.
- For a 2×2 matrix, this expression becomes:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \equiv \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \quad (3.12)$$

which reduces to
$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$which we rewrite as

$$\lambda^2 - p\lambda + q = 0 \quad (3.13)$$

where $p = a_{11} + a_{22}$ is the *Trace* of **A** and $q = \det(\mathbf{A})$.

Notes

Digression: Matrix Basics cont'd

Matrix Eigenvalues, Eigenvectors

- The roots of the quadratic equation in Eqn.(3.13) are:

$$\lambda_{1,2} = \frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2} \quad (3.14)$$

these are known as the *eigenvalues* of $\mathbf{A}$.

- Can show that any matrix $\mathbf{A}$ can be decomposed as follows:

$$\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1} \quad (3.15)$$

where $\mathbf{S}$ has *eigenvectors* of $\mathbf{A}$, $\mathbf{v}_1, \mathbf{v}_2$ on the columns,& $\mathbf{\Lambda}$ is a matrix with eigenvalues as diagonals & zeros elsewhere:

$$\mathbf{A} = \mathbf{S}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{S}^{-1} \quad (3.16)$$

Notes

Digression: Matrix Basics cont'd

Matrix Eigendecompostition, Singular Value Decomposition

- Eqn.(3.15) is useful to raise matrices to powers:

$$\mathbf{A}^3 = \left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right)\left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right)\left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right) = \left(\mathbf{S}\mathbf{\Lambda}^3\mathbf{S}^{-1}\right) \quad (3.17)$$

- The eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ are the solutions to the linear system $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for $\lambda = \lambda_1, \lambda_2$ respectively.
- As with λ's, these have physical meanings for the system.
- The process in Eqn.(3.15) is known as *eigen decomposition* for a square matrix; where the matrix is not square, it is known as *singular value decomposition*

Notes

Digression: Matrix Basics cont'd

Matrix Decomposition & Difference Equations

- For difference equations the system at time step n is related to that at the previous step $n-1$ through the system:

$$\mathbf{u}_n = \mathbf{A}\mathbf{u}_{n-1} = \mathbf{A}^n\mathbf{u}_0 \tag{3.18}$$

- Using eigendecomposition $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ and setting

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{S}^{-1}\mathbf{u}_0 = \mathbf{S}^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{n=0}$$

can see that

$$\mathbf{u}_n = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 \tag{3.19}$$

where c_1, c_2 are constants.

Notes

Digression: Matrix Basics cont'd

Matrix Decomposition, Difference & Differential Equations

- A similar result may be obtained for differential equations where the system of a second order equation (often) has a solution of the form:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}. \tag{3.20}$$

- So the solutions of difference and differential equations can be broken down into a *linear combination* of the λ's and corresponding $\mathbf{v}$'s of the original matrix system.
- This is useful in finding longterm solutions of matrix systems e.g. Fibonacci series.

Notes

Back to Fibonacci Sequences

Eigenvalues and the Fibonacci Difference Equation

- To find the long-term behaviour of Fibonacci system (Eqn.(3.5)), write (using Eqn.(3.17))

$$\mathbf{u}_n = \mathbf{A}^n \mathbf{u}_0 = \mathbf{S}\mathbf{\Lambda}^n\mathbf{S}^{-1}\mathbf{u}_0 \qquad (3.21)$$

- Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

from Eqn.(3.5), we find the characteristic equation to be

$$\lambda^2 - \lambda - 1 = 0$$

(from Eqn.(3.12)).

- This gives the eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Notes

Stability of Fibonacci Sequences

- The full eigendecomposition for $\mathbf{A}$ can then be found to be

$$\mathbf{A} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \qquad (3.22)$$

Thus Eqn.(3.21) reduces to

$$\begin{pmatrix} M_{n+1} \\ M_n \end{pmatrix} = \mathbf{S}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\mathbf{S}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (3.23)$$

- The nth Fibonacci number is 2nd element of vector on left hand side of Eqn.(3.23), M_n, can be shown to be:

$$M_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]^{0} \qquad (3.24)$$

Notes

Stability of Fibonacci Sequences cont'd

The Golden Number & Fibonacci Sequences

- $\phi = (1+\sqrt{5})/2$ is very important and was known to the Ancient Greeks as the *golden number* because rectangles with sides in the ratio $1:1.618$ were the most elegant.
- It occurs frequently in nature and persists in the everyday designs e.g. credit cards etc.
- As $\lambda_2 > 1$ & $-1 < \lambda_1 < 0$, λ_2 is λ_{max} and its magnitude means the Fibonacci sequence is monotonically increasing.
- The fact that λ_1 is negative & magnitude < 1 means it contributes a slight oscillation.

Notes

Stability of Fibonacci Sequences cont'd

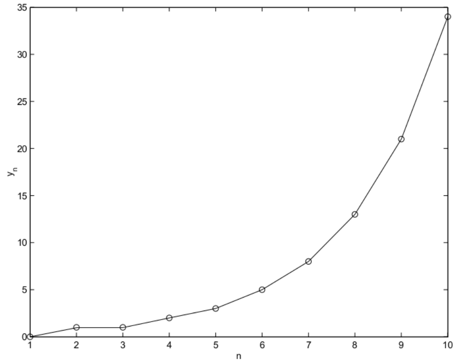

FIGURE 3.2 : Fibonacci Sequence to 10 Generations

Notes

Bonham Sequences

Example 2: Pig Reproduction

- A pair of bonhams matures to a pair of pigs next season.
- Mature pairs produce 6 pairs of bonhams the following season & every successive season thereafter
- Each pair of bonhams produced takes one season to mature & a further season to start breeding every subsequent season.
- This can be seen in the diagram (fig 3.3),
- Assume breeding is seasonal so generations do not overlap & pigs are long-lived.

Notes

Bonham Sequences cont'd

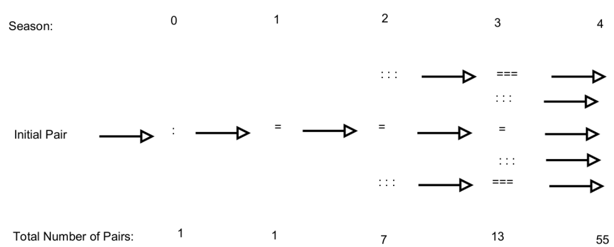

FIGURE 3.3 : Number of Immature (:) & Mature Pigs (=)

Notes

Bonham Sequences cont'd

- As with eqn(3.5), can express number of pairs of pigs in $n+1$th generation w.r.t. nth:

$$\mathbf{u}_{n+1} = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix} \mathbf{u}_n. \tag{3.25}$$

which (from eqn(3.12)) leads to the eigenvalue problem:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \equiv \begin{vmatrix} 1-\lambda & 6 \\ 1 & 0-\lambda \end{vmatrix} = 0 \tag{3.26}$$

which reduces to

$$\lambda^2 - \lambda - 6 = 0 \tag{3.27}$$

giving eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$.

Notes

Bonham Sequences cont'd

- The full eigendecomposition can then be found to be:

$$\mathbf{A} = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \tag{3.28}$$

Thus, as in eqn(3.18) above for Fibonacci:

$$\mathbf{u}_n = \mathbf{A}\mathbf{u}_{n-1} = \mathbf{A}^n\mathbf{u}_0 \tag{3.29}$$

Which may be shown to be:

$$\mathbf{u}_n = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3^n & 0 \\ 0 & -2^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\mathbf{u}_0 \tag{3.30}$$

Which reduces to

$$\mathbf{u}_n = \frac{1}{5}\begin{bmatrix} 3(3^n) + 2(-2)^n \\ 3^n - (-2)^n \end{bmatrix} \tag{3.31}$$

for an initial population $\mathbf{u}_0 = (1, 0)^T$ (i.e. one breeding pair).

Notes

A Small Wager

Example 3: A Small Wager

- A cautious but enthusiastic fan speculate on a team winning consecutive weekly matches.
- From Week 1 with €1, bet at 1.05 (i.e. $\frac{21}{20}$ odds) on the previous week's winnings plus a bet at 1.1 (i.e. $\frac{11}{10}$)on the week before's.
- Assuming fan is lucky every week, see how their winnings accumulate.
- So, following Eqn.(3.3), if amount at week $n+1$ is given by M_{n+1}:

$$M_{n+1} = 1.05M_n + 1.1M_{n-1}. \quad (3.32)$$

- With $M_0 = 0$, $M_1 = 1$

Notes

A Small Wager (cont'd)

Hence

$$\begin{aligned} 1.05M_{n+1} + 1.1M_n &= M_{n+2} \\ M_{n+1} &= M_{n+1} \end{aligned} \quad (3.33)$$

which, by writing (as per *Rabbit Reproduction* above)

$$\mathbf{u}_n = \begin{pmatrix} M_{n+1} \\ M_n \end{pmatrix}$$

takes the form

$$\mathbf{u}_{n+1} = \begin{pmatrix} 1.05 & 1.1 \\ 1 & 0 \end{pmatrix} \mathbf{u}_n, \quad \text{with} \quad \mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.34)$$

Thus $\mathbf{u}_1 = \begin{pmatrix} 1.05 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2.2025 \\ 1.05 \end{pmatrix}$ etc.

Notes

A Small Wager (cont'd)

- As with eqn(3.5) & eqn(3.25)), derive an expression for the amount in the $n+1$th week w.r.t. the nth week:

$$\mathbf{u}_{n+1} = \begin{pmatrix} 1.05 & 1.1 \\ 1 & 0 \end{pmatrix} \mathbf{u}_n. \qquad (3.35)$$

which (from eqn(3.12)) leads to the eigenvalue problem:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \equiv \begin{vmatrix} 1.05-\lambda & 1.1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \qquad (3.36)$$

which reduces to

$$\lambda^2 - 1.05\lambda - 1.1 = 0 \qquad (3.37)$$

giving eigenvalues $\lambda_1 = 1.7$ and $\lambda_2 = -0.65$.

Notes

A Small Wager (cont'd)

Thus, using Eqn(3.17) above

$$\mathbf{A}^3 = (\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1})(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1})(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}) = (\mathbf{S}\mathbf{\Lambda}^3\mathbf{S}^{-1}), \qquad (3.38)$$

the full eigendecomposition can then be found to be:

$$\mathbf{A} = \begin{pmatrix} 1.7 & -0.65 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.7 & 0 \\ 0 & -0.65 \end{pmatrix} \begin{pmatrix} 0.43 & 0.28 \\ -0.43 & 0.78 \end{pmatrix} \qquad (3.39)$$

So, as in eqn(3.18) above for the Fibonacci example:

$$\mathbf{u}_n = \mathbf{A}\mathbf{u}_{n-1} = \mathbf{A}^n \mathbf{u}_0 \qquad (3.40)$$

Notes

A Small Wager (cont'd)

Which may be shown to be:

$$\mathbf{u}_n = \begin{pmatrix} 1.7 & -0.65 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.7^n & 0 \\ 0 & (-0.65)^n \end{pmatrix} \times \begin{pmatrix} 0.43 & 0.28 \\ -0.43 & 0.78 \end{pmatrix} \mathbf{u}_0$$

Which reduces to

$$\mathbf{u}_n = 0.43 \begin{bmatrix} (1.7^{n+1}) - (-0.65)^{n+1} \\ 1.7^n - (-0.65)^n \end{bmatrix} \quad (3.41)$$

for an initial sum of $\mathbf{u}_0 = (1,0)^T$, this gives$\mathbf{u}_1 \approx \begin{pmatrix} 1.05 \\ 1 \end{pmatrix}$ etc.

Notes

The Leslie Matrix

- The *Leslie* matrix is a generalization of the above.
- It is a matrix which describes the increases in numbers in various age categories of a population year-on-year.
- As above we write $\mathbf{p}_{n+1} = \mathbf{A}\mathbf{p}_n$ where $\mathbf{p}_n$, $\mathbf{A}$ are given by:

$$\mathbf{p}_n = \begin{pmatrix} p_n^1 \\ p_n^2 \\ \vdots \\ p_n^m \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{m-1} & \alpha_m \\ \sigma_1 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \sigma_{m-1} & 0 \end{pmatrix} \quad (3.42)$$

where the Leslie matrix **A** is made up of α_i, σ_i, the number of births in a given age class i in year n & the fraction/probability that i year-olds survive to be $i+1$ years old, respectively.

Notes