CA659 Mathematical Models/Computational Science

In-Class Exercise 8

A Problem On Linear Interaction Models

1. Back when we covered Markov processes, we came across matrix models of the following type:

$$\boldsymbol{M}_{n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \boldsymbol{M}_n$$

or, to write the system another way: $x_{n+1} = ax_n + by_n$

$$y_{n+1} = cx_n + dy_n$$

Where:

$\boldsymbol{M}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ denotes the state of the system at discrete time point *n*;

$\boldsymbol{M}_{n+\mathbf{1}} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}$ denotes the state of the system at discrete time point *n+1*;

x_n, y_n; are the fractions of the population who have opted for (to take an example) Cadbury over Nestle at discrete time point *n*;

a, b, c, d; are the coefficients of the Transition matrix;

Now we will model similar systems using continuous models we represent changes over time using derivatives with respect to time and differential equations. So, if:

$x(t), y(t)$ are the fractions of the chocolate bar market occupied by Cadbury, Nestle (respectively) at time t;

$\propto_{xy}$ is the rate of change of market share from Nestle y towards Cadbury x;

$\propto_{yx}$ is the rate of change of market share[1] from Cadbury x towards Nestle y,

i. Write down the differential equations for change of market share for Cadbury (i.e. $\frac{dx}{dt}$) and Nestle (i.e. $\frac{dy}{dt}$).

ii. Hence show that: $x(t) + y(t)$=1.

iii. Show that the equilibrium points $(\bar{x}, \bar{y})$ of the system (i.e. those values $(\bar{x}, \bar{y})$ for which $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) are given by:

$$(\bar{x}, \bar{y}) = \left(\frac{\propto_{xy}}{\propto_{xy} + \propto_{yx}}, \frac{\propto_{yx}}{\propto_{xy} + \propto_{yx}} \right)$$

iv. If, initially, both Cadbury and Nestle have equal shares of the market and if the fluxes are given by $\propto_{xy} = \frac{1}{3}, \propto_{yx} = \frac{2}{3}$, find the market equilibrium values $(\bar{x}, \bar{y})$.

[1] This is also known as the "Brand Preference Flux"