

CA659 Mathematical Models/Computational Science

In-Class Exercise 6

Problems On Differential Equations

1. Write down the equation for unconstrained (Malthusian) growth. Show that the doubling time (i.e. the time for the population to double in size) is given by $t_2 = \log_e 2/r$, where r is the constant growth rate.
2. The spread of a rumour in a population can be modelled according to the equation:

$$\frac{ds}{dt} = ks[M - s] \quad (1)$$

Where:

$s(t)$, denotes those that have heard the rumour after time t ;

M , denotes the number of individuals in the population;

k , is a constant dependent on how exciting/unexpected the rumour is.

If $M = 1000$ people, show that:

$$\frac{1}{s(1000 - s)} = \frac{1}{1000s} + \frac{1}{1000(1000 - s)} \quad (2)$$

Let

$$\frac{1}{s(1000-s)} = \frac{A}{s} + \frac{B}{(1000-s)}$$

Then

$$\frac{1}{s(1000-s)} = \frac{A}{s} + \frac{B}{(1000-s)} \text{ for constant } A, B$$

So multiplying both sides by the denominator on the left hand side gives:

$$1 = A(1000 - s) + Bs$$

So, equating coefficients of s^0 gives $A = \frac{1}{1000}$

And equating coefficients of s^1 gives: $0 = -A + B$ and hence $B = \frac{1}{1000}$

Therefore $\frac{1}{s(1000-s)} = \frac{1}{1000s} + \frac{1}{1000(1000-s)}$ QED

Using the expression you have derived in Equation (2) above, and the initial condition:

$$s(0) = 500 \text{ people}$$

find the general solution to Equation (1) above, the number who have heard the rumour after time t . If $k=1$ How long will it take for three quarters of the population to have heard the rumour? (You may leave your answer in terms of logarithms).

$$\int_{500}^{s(t)} \frac{1}{s(1000-s)} ds = \int_0^t k dt$$

Therefore, from Equation (4.2)

$$\frac{1}{1000} \left[\int_{500}^{s(t)} \frac{ds}{s} + \int_{500}^{s(t)} \frac{ds}{(1000-s)} \right] = \int_0^t k dt$$

And hence:

$$\frac{1}{1000} \left[\log_e \frac{s}{(1000-s)} \right]_{500}^{s(t)} = k[t]_0^t$$

Simplifying gives:

$$\frac{1}{1000} \left[\log_e \frac{s}{(1000-s)} \right] - \frac{\log_e 1}{1000} = kt$$

With $\log_e(1)=0$ this becomes, in terms of t :

$$\frac{1}{1000} \log_e \frac{s}{(1000-s)} = kt$$

Therefore, for $k = 1$ and $s(t) = 750$ people,

$$t = \frac{1}{1000} \log_e \frac{750}{250} = \frac{\log_e 3}{1000}$$

3. Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates that the rate of change in the infected population is proportional to product of the number of mice who have the disease with the number that are disease free. Assuming that the theory is correct, and that $k = \frac{10^{-3}}{\text{hour}}$ find how long it will take half the population to contract the disease.

Let $N(t)$ denote the number of mice with the disease at time t . We are given that $N(0)=5$ mice and it follows that $500 - N$ is the number of mice without the disease at time t . The theory predicts that:

$$\frac{dN}{dt} = kN[500 - N]$$

Show that the general solution for the number of mice infected with the disease after time t is given by:

$$\frac{N}{500 - N} = \frac{1}{99} e^{500kt}$$

and hence or otherwise, determine t_{250} .