CA200 – Quantitative Analysis for Business Decisions

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Table of Contents

4. Introduction to Statistics

4.1 Overview

Statistics deal with the management and quantitative analysis of data of different types, much of it numerical. Underpinning statistical theory are the mathematics of probability; in other words, it is possible to talk about expected outcomes and how reliable these outcomes are, whether reproducible and so on.

Of interest, typically, in a statistical investigation, is

(i) The collection of data from past records, from targeted surveys, from experiment and so on. Experiments can be 'controlled' to some extent, whereas surveys are passive, although 'leading questions' may be inserted to try to obtain desired answers, which is not helful to the objectivity of the investigation, but may help the marketing!

(ii) The analysis and interpretation of the data collected: - using a range of statistical techniques

(iii)Use of the results, together with probabilities, costs and revenues to make informed decisions.

Note: Much statistical data relies on **sampling**, as we can rarely obtain information from (or on) every unit of the target population, (whatever that may be). If the population is sampled correctly – i.e. using random sampling (or one of the other variant probabilistic sampling methods), then probability rules apply and we can use these for interpretation – as above. In effect, the correctly-drawn sample is considered to be **representative** of the target population and can be used to make estimates of different features of that population.

A sample, taken without this type of rigour – e.g. a 'judgement' , 'chunk', 'quota' sample, or similarly, is not a probabilistic sample and can be analysed as descriptive of itself only and not taken to be representative of the larger group.

Starting points:

- Nature or type of data
- Grouping the data in some useful way.

4.2 Discrete or Continuous data: Variable types

Discrete Data: has distinct values, no intermediate points, so e.g. can have values, such as 0,1,2,3, , but not values, such as 1.5, 2.7, 4.35,…..

Continuous Data: consists of any values over a range, so either a whole number or a fraction, so e.g. weights 10.68Kg., 14.753kg., 16kg., 21.005kg.

Random variable (i.e. a variable which can take any chance value appropriate to the data type) is used to refer to values of the data for the measure of interest, so if X = Discrete Random Variable = No. of Days (say), X might have values 0,1,2,3,…. If Y = Continuous Random Variable = Distance in Miles (say), then is has a set of any values, e.g. 12.5, 30.02, 17.8, 20.96 etc.
These random variables thus have a set of values for any given situation, which is determined by the actual dataset that is being considered. In consequence, we can refer to the probability of the random variable having a given value (or range of values), and we can obtain its expected value and so on. Thus:

Discrete Distribution: discrete data generate a discrete frequency distribution (or more generally, a description of possible outcomes by a discrete probability distribution)

Continuous Distribution: as above, but for continuous data over an interval (continuous range of possible measures/outcomes).
<u>Note1</u>: Usually, continuous distributions are more amenable to statistical analysis. Further, where samples are *large*, it is often possible to assume continuity of values, even if data type is strictly discrete.

<u>Comment</u>: In talking about data type and nature of measurement, we should also recall how *scales* (or *levels*) *of measurement* can differ and what we can do with data of discrete and continuous types, depending on what is being measured.
SoM / LoM include *Nominal*, *Ordinal*, *Interval* and *Ratio*

Examples

Nominal -use convenient labels to differentiate categories, so e.g. M and F; e.g. holiday types, e.g. product brands etc.

Ordinal – rank has some meaning for the data, so e.g. scale of satisfaction – very dissatisfied, dissatisfied, find acceptable, satisfied, very satisfied. These might be given scores 1 to 5, but these are not measurable in a numerical sense. They give rank, but do not tell us if very satisfied is twice as happy as satisfied, for example.

(We can construct a distribution for these levels of measurement, but labelling M = 1 and F = 2 for convenience, does not give the value 1.5 a real meaning. Similarly, subtracting dissatisfied (=2) from very satisfied (= 5), does not mean that these are equivalent to a distance of 'find acceptable' apart. Merely the ranking tells us the direction of increasing/decreasing satisfaction in relation to each other.).

Interval : - scale is *quantitative,* so have equality of units, and differences and amounts have meaning. Zero is just another point on the scale so values above and below this are acceptable. For example, if we consider journey distance from given start point. A is 20 miles East from starting point, B 30 miles West, so labelling the start point zero, B as -30 and A as +20 means that A and B are 50 miles apart, A is 2/3 distance B away from start point and similarly. Another classic example is the Fahrenheit temperature scale, which includes both negative and positive values, but where for C and D the same no. of units apart as E and F, the same *difference in temperature* in degrees Fahrenheit applies.

Ratio : the ratio scale is also a quantitative scale and has equality of units. However, absolute zero is a defined number, so no negative values possible. Examples include physical measures, such as volume, height, weight, age etc.

Many standard texts include summary tables such as that below to emphasise the fundamental differences between scales (or levels) of measurement.

Scale	Categories are different	Direction of difference/ relative position	Amount of difference	Absolute Zero (no negative values)
Nominal	√			
Ordinal	√	√		
Interval	√	√	√	
Ratio	√	√	√	√

Finally, note that ***qualitative*** data deals with a description in the *observable* sense, ***quantitative*** data with measurable features. The nominal level of measurement is thus somewhat distinct from *other scales* and perhaps the simplest way of thinking of it is as *non-numeric*, so 1 and 2 or M and F are just 'labels' for gender, whichever is used.

4.3 Frequency distributions: Statistical Distributions – depiction

Frequency Distribution or (Relative Frequency)or (%Frequency) records actual data values available for a given data set.

Probability Distributions (also called Statistical Distributions) describe 'possible' data sets, sampled data , e.g. past experience information. So, samples drawn from a larger group or population have associated probabilities of how these might look.

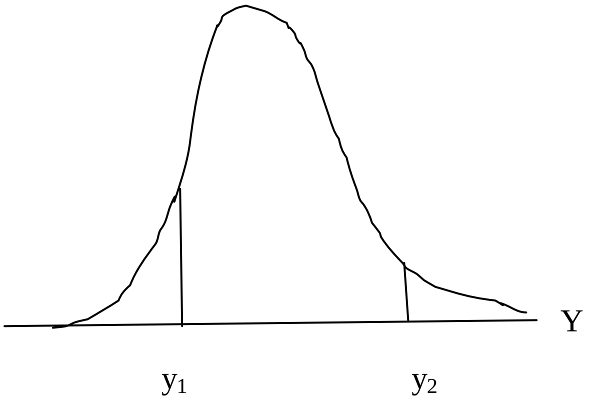
Note: Also recall that a basic definitions of (objective) probability is the long-run proportion or relative frequency, e.g. large no. coin tosees, picking a card etc.

The probability identification with relative frequency or long-run proportion is important because it means we can sub-divide or section a distribution to talk about the **% of the distribution** lying **between given values** of the variable.

Example: If Q1 value of the variable is that below which 25% of the distribution values are found, Q2 the 50% point (or median) above and below which 50% of the distribution is found, Q3 the 75% point, Q4 the 100% point or whole distribution, then Q3-Q1 defines the middle 50% or *inter-quartile range*.

Similarly, we can define other percentiles, e.g. the 10 percentile, below which 10% of the distribution lies, the 60 percentile etc.. etc.

We can similarly talk about **expecting** 25% of a **probability distribution** to lie below or above a given value of the variable or 60% of it between two given values of the variable and so on.



4.4 Characterising Distributions: (average, dispersion etc.)

Summary Statistics: Values summarising main features of the data. These are usually *representative* in some way, i.e. measures of **location** or **central tendency** or of spread (**dispersion** or **variability**)

Random Value: could pick any one in a set of data S= $\{x_1, x_2, \ldots x_n\}$, say x_k. Straightforward, but obviously extreme values can occur and successive values might be very different.

Average:

Commonly use the **Arithmetic Mean,** so for set of values, S above

$$\bar{x} = \{(x_1 + x_2 + ... + x_n)/n\}$$

or for data that are grouped, where x_1 occurs f_1 times, x_2 occurs f_2 times etc.

$$\bar{x} = \{(f_1x_1 + f_2x_2 + ... + f_nx_n)/(f_1 + f_2 + ... f_n)\}$$

$$\bar{x} = \sum f\,x / \sum f$$

Example.

For the set of class marks, given in the table, we want the **average**. Marks are presented in ranges, so need to use the midpoint of the range in order to apply the formula. All intervals are equal and there are no gaps in the classification. The range 0-19 contains marks > 0 and ≤ to 20, so mid-point taken as 10. Other intervals are similarly interpreted.

Mark Range	**Midpt. of range x_i**	**No. of students f_i**	**f_i x_i**
0-19	10	2	20
20-38	30	6	180
40-57	50	12	600
60-79	70	25	1750
80-99	90	5	450
Sum	-	50	3000

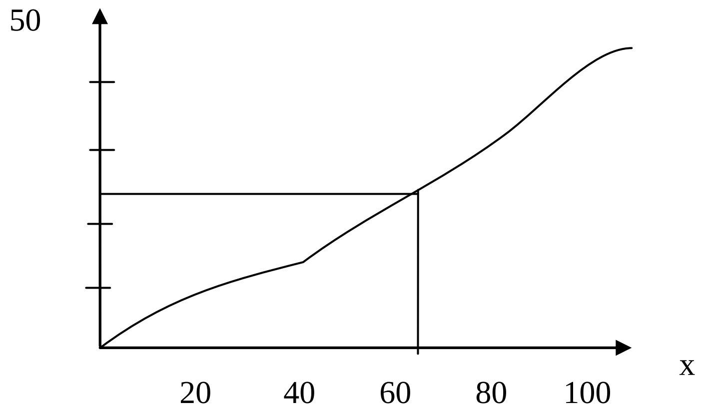
$$\bar{x} = \{3000/50\} = 60$$

Note: Population or group should be homogeneous for an average to be meaningful, so average height of all students is not representative of M or F separately.

The **Mode** is another common 'average' measure and is that value which occurs most frequently in the distribution of values.

The **Median** is the middle point of the distribution,. So, if $\{x_1, x_2, \ldots x_n\}$ are the marks of the students, placed in increasing order, the median is the mark of the *(n+1)/2* th student. This is often calculated from the **cumulative frequency** distribution.

For example, the median is the mark of the 25^{th} / 26^{th} students. From the table, this is the mark of the 5.5 th student in the 60-79 mark range, which contains 25 students and spans 20 marks so Median = 60 + [(5.5)/25] ×20 = 64.4 marks. From the diagram, it is the 50% point in the cumulative frequency diagram of marks ≥ 0 (the more than cumulative frequency) or ≤100, (the less than cumulative frequency).



Similar concepts apply for continuous distributions. The distribution function or cumulative distribution function is defined by:

$$F(x) = P\{X \le x\}$$

Its derivative is the frequency distribution or frequency function (or prob. density fn.)

$$f(x) = df(x)/dx$$

i.e.

$$F(x) = \int f(x)dx$$

Dispersion or Spread

An 'average' alone may not be adequate, as distributions may have the same arithmetic mean, but the values may be grouped very differently. For a distribution with *small variability* or *dispersion*, the distribution is concentrated about the mean. For *large variability*, the distribution may be very scattered.

The average of the squared deviations about the mean is called the **variance** and this is the commonly used statistical measure of dispersion. The square root of the variance is called the **standard deviation** and has the same units of measurement as the original values.

To illustrate why the variance is needed, consider a second set of student marks and compare with the first. Both have the same mean, but the variability of the second set is much higher, i.e. marks are much more dispersed or scattered about the mean value.

Example:

First Set of Marks					**Second Set of Marks**			
f	**x**	**fx**	**fx²**		**f**	**x**	**fx**	**fx²**
2	10	20	200		6	10	60	600
6	30	180	5400		8	30	240	7200
12	50	600	30000		6	50	300	15000
25	70	1750	122500		15	70	1050	73500
5	90	450	40500		15	90	1350	121500
50		**3000**	**198600**		**50**		**3000**	**217800**

$$\sigma^2 = Var(x) = \sum f \times squared\ \ deviations / \sum f$$

$$\sigma^2 = Var(x) = \left[\frac{\sum_i f(x_i - \bar{x})^2}{\sum f}\right] = \left[\frac{\sum_i f(x_i)^2}{\sum f}\right] - \bar{x}^2$$

$$S.D. = \sigma = \sqrt{Var(x)}$$

Then for set 1:

Var(x) =198600/50 – (60)^2 = 372 marks^2 i.e. s.d. = 19.3 marks

for set 2:

Var(x) = 217800/50 – (60)^2 = 756 marks^2 i.e. s.d. = 27.5 marks

Other Summary Statistics, characterising distributions

Average Measures (other than Arithmetic Mean): there are a number of these, apart from the simple ones of the mode and median.

Other Dispersion Measures

Range = Max. value – Min. value

Inter-quartile Range = Q3 –Q1 (as before)

Skewness

Degree of symmetry. Distributions that have a large tail of outlying values on the right-hand-side are called positively skewed or skewed to the right. Negative skewness refers to distributions with large left-hand tail.

A simple formula for skewness is

Skewness = (Mean - Mode) / Standard Deviation

so, for Set 1 of student marks:

Skewness = (60 - 67.8) / 19.287 = - 0.4044.

Kurtosis

-measures how peaked the distribution is; 3 broad types ate Leptokurtic - high peak, mesokurtic, platykurtic – flattened peak.

Characterising Probability (Statistical) Distributions:

We have seen examples of this already in the work on expected values and decision criteria.

<u>Example:</u> X = random variable = Distance in km. travelled by children to school. Past records show probability distribution to be as below. Calculations are similar.

p_i	x_i	$p_i x_i$	$p_i x_i^2$
0.15	2.0	0.30	0.60
0.40	4.0	1.60	6.40
0.20	6.0	1.20	7.20
0.15	8.0	1.20	9.60
0.10	10.0	1.00	1.00
1.00		**E(x) = $\sum$ = 5.30**	**33.80**

Then Expected Value of X = E(X) = 5.30 km.

Variance of X = Var(X) = 33.80 - $(5.30)^2$ = 5.71 km.^{2}

Standard Probability (or Statistical) Distributions

For a summary of commonly used statistical distributions – see any statistical textbook.

Importantly, characterised by expectation and variance (as for random variables) and by the parameters on which these are based.

E.g.for a Binomial distribution, the parameters are *p* the probability of success in an individual trial and *n* the No. of trials.

The probability of success remains *constant* – otherwise, another distribution applies.

Use of the correct distribution is core to statistical *inference* – i.e. estimating population values on the basis of a (correctly drawn, probabilistic) sample.

The sample is then **representative** of the population- (sampling theory)

4.5 Standard Continuous Distns. (pdfs): Normal (& Tables)

Fundamental to statistical inference (Section 5) is the Normal (or Gaussian), with parameters, μ the **mean** (or more formally) **expectation** of the distribution) and σ (or σ^2), the Standard deviation (Variance) respectively. A random variable X has a Normal distribution with mean μ and standard deviation σ if it has density

$$f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] & for \quad -\infty < x < \infty \\ & and \quad 0 \quad otherwise \end{cases}$$

<u>Notes:</u>

The Normal distribution is also the so-called 'limiting' distribution of the **Mean** of a set of independent and identically distributed random variables: this means that if you have a distribution of values for a random variable X, then whatever the distribution for X looks like, the distribution of $\overline{X}$ is Normal; $\overline{X}$ ~ N, providing that the sample size is sufficiently large.

This result is called the **Central Limit Theorem** and plays a fundamental role in sampling theory and statistical inference, as it means that the Normal distribution can often be used to describe values of a variable or its mean or proportion.

Further, the Normal can also be used to **approximate** many **empirical** data. Default $\therefore$ in much empirical work is to assume observations are distributed (~) approx. as N.

It can also be used to **approximate discrete distributions** if the sample size is large.

Tables:

Given that a Normal distribution can be generated for any combination of mean (μ) and variance (σ^2), this implies very many tables would be needed to be able to work out probabilities of (say) % of a distribution between given values, unless we wished to calculate these from first principles! Fortunately, a simple transformation means that Standardised Normal Tables can be used for **any** value of the mean and standard deviation (or variance).

X is said to be a **Standardised Normal Variable, (U),** if $\mu = 0$ and σ^2 (or σ) = *1*.

[For Normal Tables, (i.e. Standardised Normal Tables, see pages 13 sand 14 in White, Yeats and Skipworth, "Tables for Statisticians."]

4.6 Standard Discrete Distributions

Importance

Modelling practical applications (such as counts, binary choices etc.)

Mathematical properties known

Described by few parameters, which have natural interpretations.

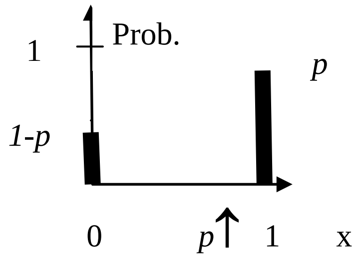
Bernoulli Distribution.

This is used to model a single trial or sample, which gives rise to just two outcomes: e.g. male/ female, 0 / 1.

Let p be the probability that the outcome is **one** and $q = 1 - p$ that the outcome is **zero**, then have:

$$E[X] = p\,(1) + (1-p)\,(0) = p$$

$$VAR[X] = p\,(1)^2 + (1-p)\,(0)^2 - E[X]^2 = p\,(1-p).$$



Binomial

Suppose that we are interested in the number of successes **X** in a number ‘n’ of **independent** repetitions of a Bernoulli trial, where the probability of success in an **individual** trial is constant and = p. Then

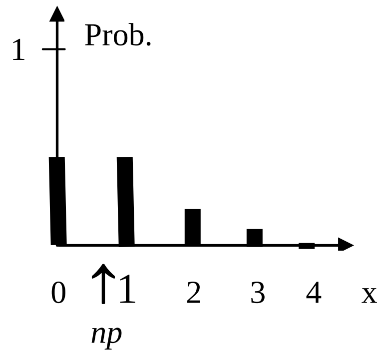
$$Prob\,\{X = k\} = {}^{n}C_{k}\, p^{k}\, (1-p)^{n-k}, \quad (k = 0, 1, \ldots, n)$$

$$E[X] = n\,p$$

$$VAR[X] = n\,p\,(1-p)$$

where Binomial Coefficients are the No. combinations of n items, taken k at a time.

$${}^{n}C_{k} \quad also \quad written \quad \binom{n}{k}$$



The shape of the distribution depends on the value of n and p.

As p → ½ and n increases, it becomes increasingly symmetric

(and can be approximated by the Normal for large n)

Poisson Distribution

The Poisson distribution = limiting case of the Binomial distribution, where n$\rightarrow\infty$, $p\rightarrow 0$ in such a way that $np\rightarrow\lambda$, the Poisson parameter

$$P\{X=k\}=e^{-\lambda}\lambda^k/k! \qquad k=0,1,2,......$$

$$E(X)=\lambda$$

$$Var(X)=\lambda$$

The Poisson is used to model the No.of occurrences of a certain phenomenon in a fixed period of time or space, e.g. people arriving in a queue in a fixed interval of time, e.g. No. of fires in given period, See insurance example Section 3B). we can therefore think of the Poisson as the *rare event* Binomial, i.e. *p* small. Its shape depends on the value of p and n (i.e. value of λ) ; usually right –skewed (*positive-skewed)* for *p small*, *n small*,

<u>Note:</u>

Tables for the **Cumulative Binomial probabilities** for a range of *n* and *p* values are given on pages 1-6 of the White, Yeats and Skipworth "Tables for Statisticians".

e.g. for *p*=0.02 and *n*=9 have:

	p=0.02		
n=9, r=0 = P{r ≥0}	1.00000		
		P{r =0}	1.00000-0.16625 =0.83375
1 = P{r ≥1}	0.16625		
		P{r =1}	0.16625-0.01311 = 0.15314
2 = P{r ≥2}	0.01311		
		P{r = 2}	0.01311-0.00061 = 0.0125
3 = P{r ≥3}	0.00061		
		P{r = 3}	0.00061-0.00002 =0.00059
4 = P{r ≥ 4}	0.000002		

Tables for the Cumulative Poisson probabilities for a range of λ values are given on pages 7-10 of the White, Yeats and Skipworth “Tables for Statisticians” and probabilities for exact values 0,1,2, etc. are calculated as for the Binomial example above.

<u>Example:</u>

Probability that a salesman makes a sale on a visit to a prospective customer is 0.2. What is prob., in 2 visits of

(i) making no sales? (ii) making 1 sale? (iii) making 2 sales?

<u>Solution.</u>

Let p = prob. sale = 0.2 and q = probability no sale = 0.8. So various combinations can be tabulated.

Visit 1	**Visit 2**	**Probabilities**
Sale	Sale	$p \times p = 0.04$
Sale	No sale	$p \times q = 0.2 \times 0.8 = 0.16$
No Sale	Sale	$q \times p = 0.8 \times 0.2 = 0.16$
No Sale	No Sale	$q \times q = 0.8 \times 0.8 = 0.64$

So P(No Sales) = 0.64, P(1 Sale) = 0.32, P (2 Sales) = 0.04

[Digression : Clearly, more work is involved if we have a larger no. visits (trials or sample size) but general form can be used: $(p+q)^n$

Where p = prob. success , q = probability failure in n trials (or in sample of size n).

So for example,

$(p+q)^2 = p^2 + 2pq + q^2 = (0.2)^2 + 2(0.2 \text{x} 0.8) + (0.8)^2 = 1.0$ as before, with coefficients of terms being 1,2 and 1.

Clearly for $(p+q)^3$ have $p^3 + 3p^2q + 3pq^2 + q^3$ with coefficients 1, 3, 3, 1

for $(p+q)^4 = p^2 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$ with coefficients 1, 4, 6, 4, 1 etc.

[Gives Pascal's triangle for coefficients of binomial expansion, which you may recall from way back?]

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 etc.]

Grows very fast, however, so usually more efficient to use the 'no. of combinations' expression, that we have had before, directly for the coefficients of the probability terms, i.e.

$$^{n}C_{k} \quad or \left[\binom{n}{k}\right]$$

So, e.g. $^5C_0 = 1$, $^5C_1 = 5$, $^5C_2 = 10$, $^5C_3 = 10$, $^5C_4 = 5$, $^5C_5 = 1$

Obviously, do not need **whole** expansion if only **one value** required, so if probability of success =0.8, probability of failure = 0.2, then

P{10 successes in sample, size 12} = $^{12}C_{10}\, p^{10} q^{2}$ = [12! /(10!)(2!)] × $(0.8)^{10}$ $(0.2^{2}$

4.7 Exercises:

These will be specified separately